Introduction to Mathematical Methods of Analytical Mechanics

Henri Gouin

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First published 2020 in Great Britain and the United States by ISTE Press Ltd and Elsevier Ltd

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British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
Library of Congress Cataloging in Publication Data
A catalog record for this book is available from the Library of Congress
ISBN 978-1-78548-315-8

Printed and bound in the UK and US

Preface

The objective of this book is to offer an overview of geometric methods of calculus of variations and how this can be applied in analytical mechanics. It is followed by the study of properties of spaces in mechanical systems with a finite number of degrees of freedom. The book was inspired, in part, by methods proposed by Pierre Casal, a former Professor at the Faculty of Sciences of Marseilles University. These methods were considered again and developed during a course that I taught to students in the third year of the Applied Mathematics Bachelor's program.

The mathematical tools used throughout the book are those used in elementary algebra, analysis and differential geometry. The book does not require mathematical tools that would be beyond the scope of a third-year university student (readers may refer, among others, to the works of (Queysane 1971, Couty and Ezra 1980, Martin 1967, Brousse 1968).

Part 1

In the first part of the book, we present geometric methods used in the calculus of variations. Free extrema or extrema that are related to integral or non-integral constraints are studied. These make it possible to introduce the concept of the *Lagrange* multiplier. An initial study of the *Hamilton equations*[1] associated with the concept of a generating function follows from these methods. Research into the geodesics of surfaces is also a natural application that uses differential geometry. These methods of differential geometry can be extended to the calculation of the variation of curvilinear integrals. Two forms of variations can be made and explicitly discussed: the first form uses the concept of variation of a vectorial derivative; the second form is similar to finding the optical path followed by light in a medium with

1 It should be noted that these equations are likely to have been written by *Huygens*, and the symbol H corresponds to his initials.

a variable refractive index. This leads to the study of *Descartes' laws* and isoperimetric problems. *Noether's theorem*, groups of invariance associated with differential equations and the concept of a *Lie group* are natural extensions of these calculations. Thus, *Fermat's principle*, associated with the optical path followed by light, leads to finding first integrals. The tools used are related to tensor calculations that bring in the structure of a vector space and its dual space.

Part 2

The second part of the book presents the application of the tools discussed earlier to the mechanics of material systems with a finite number of degrees of freedom. After briefly reviewing the *d'Alembert principle*, we introduce the concept of Lagrangian defined in space-time by the homogeneous Lagrangian. We find that the first integral in mechanics is associated with Noether's theorem. The reintroduction of partial results leads to the *Maupertuis principle* and to the introduction of *Riemannian geometry* in the case of the conservation of energy; this is the foundation for *de Broglie wave mechanics*. The integration methods for equations in mechanics are analyzed using the *Jacobi method* and its application in the important case of *Liouville's integrability*. This section continues with the concepts of *angular variables* and *action variables* for periodic and multi-periodic motions, presented by *Delaunay*, which are especially useful in celestial mechanics. The second part concludes with the study of space in analytical mechanics, including various phase spaces. The concepts of *dynamic variables*, *Lie brackets*, *Poisson brackets* and *Lie algebra* are natural developments of this study. We then briefly return to first integrals when studying Poisson brackets with two dynamic variables. Canonical transformations that conserve the form of the mechanical equations in different dynamic variables lead to the concept of *symplectic scalar product*, which is the first step in studying *symplectic geometry*.

Part 3

The third part presents some easy applications of differential equations to mechanical systems. The concept of *flow* in phase space leads to the *Liouville theorem*, which is essential in statistical mechanics. It corresponds to the conservation of volume in this space and can be interpreted using the *Poincaré recurrence theorem*. The small motions of mechanical systems are analyzed using the specific case of the *Weierstrass discussion*. The equilibrium positions of the systems associated with autonomous differential equations bring us to the concepts of *Lyapunov stability* and *asymptotic stability*. The necessary conditions for stability are presented in the context of the *Lejeune Dirichlet theorem*. The concept of linearization of differential equations in the neighborhood of an equilibrium position makes it possible to study the small oscillations of dynamic Lagrangian systems and their frequency, as well as the systems' disturbances. This part ends with a discussion

on the stability of periodic systems and the topology of phase spaces, with the extension of the concepts of Lyapunov stability, asymptotic stability and the new concept of *strong stability* in the neighborhood of an equilibrium position. The *Hill* and *Mathieu* equations allow us to apply these concepts.

At the end of the book, the reader can find a collection of exercises for geometrical and mechanical applications, followed by bibliographical references and an index.

I would like to thank Françoise for reviewing the proofs and helping me with her valuable comments.

Henri GOUIN
October 2019

Mathematicians, Physicists and Astronomers Cited in this Book

Jean le Rond d'Alembert (1717–1783), French mathematician, physicist and philosopher.

Vladimir Igorevitch Arnold (1937–2010), Russian mathematician.

Louis de Broglie (1892–1987), French mathematician and physicist.

Charles-Eugène Delaunay (1816–1872), French mathematician.

René Descartes (1596–1650), French mathematician, physicist and philosopher.

Pierre de Fermat (1601–1665), French mathematician.

William Hamilton (1805–1865), English–Irish physicist and astronomer.

George Hill (1838–1914), American mathematician and astronomer.

Christiaan Huygens (1629–1695), Dutch mathematician.

Charles Jacobi (1804–1851), German mathematician.

Johann Lejeune Dirichlet (1805–1859), German mathematician.

Sophius Lie (1842–1899), Norwegian mathematician.

Joseph Liouville (1809–1882), French mathematician.

Aleksander Mikhailovitch Lyapunov (1857–1918), Russian mathematician.

Emile Mathieu (1835–1890), French mathematician.

Pierre de Maupertuis (1698–1759), French mathematician, physicist, astronomer and naturalist.

Amalie Emmy Noether (1882–1939), German mathematician.

Henri Poincaré (1854–1912), French mathematician, physicist and philosopher.

Siméon Denis Poisson (1781–1840), French mathematician and physicist.

Bernhard Riemann (1826–1866), German mathematician.

Karl Weierstrass (1815–1897), German mathematician.

Important Notations

$\star$	: transpose operation in a vector space
$\boldsymbol{x}, \boldsymbol{Q} \cdots$	: vectors in a vector space, represented in bold italics
$\boldsymbol{x}^\star, \boldsymbol{Q}^\star \cdots$	: linear transposed forms of the vectors in a vector space
$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$	: n-tuples of $\mathbb{R}^n$ written as columns of the elements of the vectors $\boldsymbol{x}$, $\boldsymbol{Q}$ written in the canonical basis
$[x_1, \cdots, x_n],\ [q_1, \cdots, q_n]$	: n-tuples of $\mathbb{R}^{n\star}$ written as rows of the linear forms of the components $\boldsymbol{x}^\star$, $\boldsymbol{Q}^\star$, where $\mathbb{R}^{n\star}$ denotes the dual of $\mathbb{R}^n$
$\dfrac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}$	: linear mapping $\mathbb{R}$ defined by the relation $d\boldsymbol{V} = \dfrac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\, d\boldsymbol{x}$ and represented by the matrix $\begin{bmatrix} \dfrac{\partial v_1(x_1,...,x_n)}{\partial x_1}, \cdots, \dfrac{\partial v_1(x_1,...,x_n)}{\partial x_n} \\ \vdots \\ \dfrac{\partial v_n(x_1,...,x_n)}{\partial x_1}, \cdots, \dfrac{\partial v_n(x_1,...,x_n)}{\partial x_n} \end{bmatrix}$, where $\boldsymbol{V}$, a function of $\boldsymbol{x}$, is represented by the column matrix $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\dot{a} = \dfrac{da}{dt}$	: derivative of a with respect to time in Newtonian notation. Similarly, $\dot{\boldsymbol{Q}} = \dfrac{d\boldsymbol{Q}}{dt}$
$\mathbf{1}$	: identity tensor in a vector space
grad	: gradient in a vector space
rot	: rotational in $\mathbb{R}^3$

1

Elementary Methods to the Calculus of Variations

The calculus of variations is done using all methods that allow the resolution of extremum problems. Numerous problems in physics can be solved using variational methods. In mechanics, for example, an *equilibrium position* is one where the potential of forces applied to the considered system is an extremum. In optics, the *optical path* followed by light is an extremum. In capillarity, *the surfaces of bubbles and drops* whose volume is given is the value that makes them minimal. We will see that a non-dissipative motion is one that makes the *Hamiltonian action* an extremum.

The problems that we study are introduced in different forms:

– In numerical form: the unknown consists of a set of scalars or functions. When the unknown is a set of scalars or functions, the calculus of variations is carried out using elementary differential calculus. This is the case for n scalars $\boldsymbol{x} = (x_1, ..., x_n)$, which is an element of $\mathbb{R}^n$ or for functions of the form

$$t \in [t_1, t_2] \subset \mathbb{R} \longrightarrow \phi(t) \in \mathbb{R}^n.$$

– In geometric form: the unknown is represented by a set of points, curves or surfaces.

1.1. First free extremum problems

The unknown is composed of n scalars $\boldsymbol{x} = (x_1, ..., x_n) \in \mathbb{R}^n$. We will determine the values of $\boldsymbol{x}$ such that $a = G(\boldsymbol{x})$ becomes an extremum, where G, assumed to be differentiable, is a mapping from $\mathbb{R}^n$ to a real set $\mathbb{R}$. The reasoning used is as follows:

let us write $d\boldsymbol{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ an n-tuple of $\mathbb{R}^n$, which we name the *variation* of $\boldsymbol{x}$. We can derive the value of the differential of $G(\boldsymbol{x})$,

$$da = \frac{\partial G}{\partial x_1}(x_1, \cdots, x_n)\, dx_{_1} + ... + \frac{\partial G}{\partial x_n}(x_1, \cdots, x_n)\, dx_n,$$

which can be written in matrix form as

$$da = \left[\frac{\partial G}{\partial x_1}(x_1, \cdots, x_n), \cdots, \frac{\partial G}{\partial x_n}(x_1, \cdots, x_n)\right] \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}.$$

The column matrix $\begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ represents a vector in $\mathbb{R}^n$ given by its elements in the canonical basis of $\mathbb{R}^n$, $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \boldsymbol{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$. The vectors will not be written with an arrow, but in bold italic letters. The line matrix

$$\left[\frac{\partial G}{\partial x_1}(x_1, \cdots, x_n), \cdots, \frac{\partial G}{\partial x_n}(x_1, \cdots, x_n)\right]$$

represents a linear form, element of the dual space of $\mathbb{R}^n$, denoted by $\mathbb{R}^{n\,\star}$ (the dual space is also designated by $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$). This linear form is expressed in the dual basis

$$\boldsymbol{e}_1^\star = [1, \cdots, 0]\ , \ldots, \boldsymbol{e}_n^\star = [0, \cdots, 1]$$

of the basis $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ and satisfies $\boldsymbol{e}_i^\star\ \boldsymbol{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, by a line matrix with n columns. Thus, * designates the *transposition* in $\mathbb{R}^n$ assumed to be Euclidean. We have

$$G'(\mathbf{x}) = \frac{\partial G}{\partial x_1}(x_1, \cdots, x_n)\ \boldsymbol{e}_1^\star + \cdots + \frac{\partial G}{\partial x_n}(x_1, \cdots, x_n)\ \boldsymbol{e}_n^\star$$

and

$$d\boldsymbol{x} = dx_{_1}\boldsymbol{e}_1 + \cdots + dx_n\, \boldsymbol{e}_n,$$

and it is possible to write

$$da = G'(\boldsymbol{x})\, d\boldsymbol{x},$$

which is called the variation of a. For reasons that will be understood later in the book, we write $\delta\boldsymbol{x}$ and δa instead of $d\boldsymbol{x}$ and da, respectively.

DEFINITION 1.1.– a is an extremum at $\boldsymbol{x}$ if and only if, for any variation $\delta\boldsymbol{x}$, the variation δa is zero. This assertion can be written as

$$\forall\, \delta\boldsymbol{x},\, \delta a \equiv G'(\boldsymbol{x})\, \delta\boldsymbol{x} = 0$$

and as a result, $G'(\boldsymbol{x}) = \mathbf{0}^\star$, where G' is a linear form of $\mathbb{R}^{n\star}$, which can be developed as

$$\frac{\partial G}{\partial x_1}(x_1, \cdots, x_n) = 0, \cdots, \frac{\partial G}{\partial x_n}(x_1, \cdots, x_n) = 0.$$

The definition is a common one in the calculus of variations. It is, nevertheless, preferable to call it a *stationary point* instead of an *extremum*, as it will be seen in the simple example given below.

EXAMPLE 1.1.– Consider the mapping $(x, y) \in \mathcal{O} \subset \mathbb{R}^2 \longrightarrow f(x, y) \in \mathbb{R}$ where $\mathcal{O}$ is an open set of $\mathbb{R}^2$. It is assumed that at $(x_0, y_0) \in \mathcal{O}$, $f(x, y)$ is an extremum.

Without loss of generality, it can be assumed that $(x_0, y_0) = (0, 0)$ and $f(x_0, y_0) = 0$. The point $(0, 0)$ corresponds to a maximum of $f(x, y)$ if and only if there exists $r \in \mathbb{R}$ such that for any (x, y) satisfying $0 \leq x^2 + y^2 \leq r^2$, we have $f(x, y) \leq 0$. Similarly, the point $(0, 0)$ will correspond to a minimum of $f(x, y)$ if and only if there exists $r \in \mathbb{R}$ such that for any (x, y) satisfying $0 \leq x^2 + y^2 \leq r^2$, we have $f(x, y) \geq 0$. Assume that f belongs to the C^2 class; in the neighborhood of $(0, 0)$, then $f(x, 0)$ and $f(0, y)$ are extrema at $(0, 0)$. Hence, $f'_x(0, 0) = 0$ and $f'_y(0, 0) = 0$. If these conditions are satisfied, the second-order Taylor–Young formula implies

$$f(x, y) = \frac{1}{2}\left(\alpha\, x^2 + 2\,\beta\, xy + \gamma\, y^2\right) + \left(x^2 + y^2\right)\varepsilon(x, y),$$

where $\lim \varepsilon(x, y) = 0$ when $(x, y) \longrightarrow (0, 0)$, with

$$\alpha = \frac{\partial^2 f}{\partial x^2}(0, 0),\ \beta = \frac{\partial^2 f}{\partial x \partial y}(0, 0),\ \gamma = \frac{\partial^2 f}{\partial y^2}(0, 0).$$

Consequences: if $\beta^2 - \alpha\gamma < 0$, we have a minimum when $\alpha > 0$ and a maximum when $\alpha < 0$; if $\beta^2 - \alpha\gamma > 0$, the quadratic form $\alpha\, x^2 + 2\,\beta\, xy + \gamma\, y^2$ is not defined, and there is no extremum for $f(x, y)$, but there is a saddle point. If $\beta^2 - \alpha\gamma = 0$, we cannot arrive at any conclusion; for example, in the case where all second derivatives are zero at $(0, 0)$ and if a third derivative is non-zero, there is neither a maximum nor a minimum. It is still important to note that the definition for an extremum is associated with the condition $G'(\boldsymbol{x}) = \mathbf{0}^\star$, which in all cases is called the stationarity condition.

EXAMPLE 1.2.– Consider in a plane three points A, B and C. We wish to find a point M in the plane such that $l(M) \equiv MA + MB + MC$ is of minimum length.

Given that $l(M)$ is continuous and greater than zero, the length $l(M)$ does admit a lower limit. Given that it is possible to limit ourselves to a compact domain in the plane (such as the domain bounded by a disk whose radius is sufficiently large), the minimum is obtained.

Let us calculate the variation denoted by $\delta l(M)$: $\delta l(M) \equiv \delta MA + \delta MB + \delta MC$. We first compute δMA; let us carry out the calculus using orthonormal axes whose origin is A; for M with the coordinates x and y, $r = MA = \sqrt{x^2 + y^2}$. We derive

$$\delta MA = \frac{x}{\sqrt{x^2 + y^2}}\,\delta x + \frac{y}{\sqrt{x^2 + y^2}}\,\delta y = \mathrm{grad}^\star\, r \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$$\text{where } \mathrm{grad\,r} = \boldsymbol{u}_A \text{ with } \boldsymbol{u}_A = \frac{\boldsymbol{AM}}{\|\boldsymbol{AM}\|}.$$

The bipoint $\boldsymbol{AM}$ corresponds to the vector of tail A and head M. The computation is carried out in the same way for the points B and C, and we obtain

$$\delta l(M) = 0 \iff \boldsymbol{u}_A + \boldsymbol{u}_B + \boldsymbol{u}_C = \mathbf{0}.$$

The point M belongs to the arc that is able to support AB with the angle $\frac{2}{3}\pi$. This is the same for BC and for AC. In order for the arcs to have a common point, it is essential that $|\widehat{ABC}| < \frac{2}{3}\pi$; no angle in triangle ABC can be larger than $\frac{2}{3}\pi$. The process for finding the point M is represented in Figure 1.1.

In the case where one of the vertices in triangle ABC can have an angle greater than $\frac{2}{3}\pi$, the point M cannot be found at any point in the plane except A, B or C. Indeed, we know that an extremum exists and that A, B and C are the points for which $l(M)$ is not differentiable; at these points, the calculus given above is not valid. It must also be noted that there exists a single solution and this single solution is a minimum.

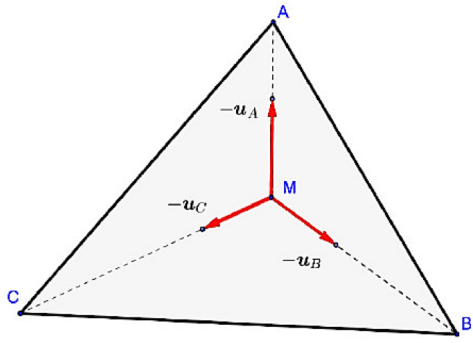


Figure 1.1. *Representation of the triangle ABC in the case where no vertex can have an angle greater than or equal to $\frac{2}{3}\pi$. The point M corresponds to the minimum distance $MA + MB + MC$. For a color version of the figures in this chapter, see www.iste.co.uk/gouin/mechanics.zip*

1.2. First constrained extremum problem – Lagrange multipliers

1.2.1. *Example of Lagrange multiplier*

THEOREM 1.1.– Let $\mathcal{A}$ be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^q$ and let $\mathcal{B}$ be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$. Regardless of the vector $\boldsymbol{V}$ in $\mathbb{R}^n$, $\mathcal{B}\boldsymbol{V} = 0$ implies $\mathcal{A}\boldsymbol{V} = 0$ is equivalent: there exists a linear mapping Λ, from $\mathbb{R}^p$ to $\mathbb{R}^q$, called the Lagrange multiplier such that $\mathcal{A} = \Lambda\mathcal{B}$.

PROOF.– As the property is an equivalence, it must be formulated as a necessary and sufficient condition.

$\Leftarrow$ If $\boldsymbol{V} \in \operatorname{Ker}\mathcal{B}$, then $\Lambda\mathcal{B}\boldsymbol{V} = 0$ and $\boldsymbol{V} \in \operatorname{Ker}\mathcal{A}$.

$\Rightarrow$ Reciprocally, assume that $\operatorname{Ker}\mathcal{A} \subset \operatorname{Ker}\mathcal{B}$, $\operatorname{Ker}\mathcal{A}$ being a vector sub-space of $\mathbb{R}^n$. There exists an additional vector sub-space E of $\operatorname{Ker}\mathcal{B}$ such that $\mathbb{R}^n = \operatorname{Ker}\mathcal{B} \bigoplus E$. Thus, $\mathcal{B}_{|E}$ is an isomorphism from E to $\operatorname{Im}\mathcal{B} \subset \mathbb{R}^p$ and is invertible; and $\left(\mathcal{B}_{|E}\right)^{-1}$ is a linear mapping from $\operatorname{Im}\mathcal{B}$ to E. Let us write $\Lambda_1 = \mathcal{A}\left(\mathcal{B}_{|E}\right)^{-1}$, a mapping from $\operatorname{Im}\mathcal{B}$ to $\mathbb{R}^q$. We can write $\mathbb{R}^p = \operatorname{Im}\mathcal{B} \bigoplus E'$, where E' is an additional vector space of $\operatorname{Im}\mathcal{B}$ relative to $\mathbb{R}^p$. Let Λ_2 be an arbitrary linear mapping over E' (we can consider, for example, the linear mapping with zero values). Thus,

$$\forall\, \boldsymbol{V}, \boldsymbol{V} \in \mathbb{R}^p, \boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2$$

This decomposition is unique. Let us posit that Λ is a linear mapping from $\mathbb{R}^p$ to $\mathbb{R}^q$ such that $\Lambda(\boldsymbol{V}) = \Lambda_1(\boldsymbol{V}_1) + \Lambda_2(\boldsymbol{V}_2)$. Therefore, $\mathcal{A} = \Lambda\mathcal{B}$. Indeed, for any $\boldsymbol{W}$, being a vector in $\mathbb{R}^n$, it is possible to write $\boldsymbol{W} = \boldsymbol{W}_1 + \boldsymbol{W}_2$, where $\boldsymbol{W}_1 \in \operatorname{Ker}\mathcal{B}$ and $\boldsymbol{W}_2 \in E$, the decomposition being unique. On the one hand, $\mathcal{A}\left(\mathcal{B}_{|E}\right)^{-1}\mathcal{B}(\boldsymbol{W}_1 + \boldsymbol{W}_2) = \mathcal{A}(\boldsymbol{W}_2)$ since $\left(\mathcal{B}_{|E}\right)^{-1}\mathcal{B}$ is the identity over E and $\mathcal{B}(\boldsymbol{W}_1) = 0$. On the other hand, $\mathcal{A}(\boldsymbol{W}) = \mathcal{A}(\boldsymbol{W}_1 + \boldsymbol{W}_2) = \mathcal{A}(\boldsymbol{W}_1) + \mathcal{A}(\boldsymbol{W}_2)$ and $\operatorname{Ker}\mathcal{A} \subset \operatorname{Ker}\mathcal{B}$ implies for any $\boldsymbol{W}_1 \in \operatorname{Ker}\mathcal{A}$, $\mathcal{A}(\boldsymbol{W}) = \mathcal{A}(\boldsymbol{W}_2)$. Finally, for any vector $\boldsymbol{W}$ in $\mathbb{R}^n$, $\mathcal{A}(\boldsymbol{W}) = \Lambda\mathcal{B}(\boldsymbol{W})$, i.e. $\mathcal{A} = \Lambda\mathcal{B}$. ■

1.2.2. *Application to the constrained extremum problem*

Let G be a differentiable mapping from $\mathbb{R}^n$ to $\mathbb{R}$ and F be another differentiable mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$. We wish to find values for the element $\boldsymbol{x}$ belonging to $\mathbb{R}^n$ such that $a = G(\boldsymbol{x})$ be an extremum, knowing that $\boldsymbol{x}$ satisfies the condition $F(\boldsymbol{x}) = \boldsymbol{0}$ (called *a constraint*). We must write: $\delta a = 0$ not for all $\delta\boldsymbol{x}$ but for all $\delta\boldsymbol{x}$ verifying the condition $F'(\boldsymbol{x})\,\delta\boldsymbol{x} = 0$. We write,

$$\forall\, \delta\boldsymbol{x},\ \delta\boldsymbol{x} \in \mathbb{R}^n, \quad F'(\boldsymbol{x})\,\delta\boldsymbol{x} = \boldsymbol{0} \quad \Longrightarrow \quad G'(\boldsymbol{x})\,\delta\boldsymbol{x} = 0$$

with

$$F(\boldsymbol{x}) = 0,$$

On the basis of $\mathbb{R}^n$ and $\mathbb{R}^p$, let us write F using column matrices

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \longrightarrow F(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, ..., x_n) \\ \vdots \\ f_p(x_1, ..., x_n) \end{bmatrix} \in \mathbb{R}^p$$

$$\text{and} \quad F(\boldsymbol{x}) = 0 \quad \Longleftrightarrow \quad \begin{cases} f_1(x_1, ..., x_n) = 0 \\ \vdots \\ f_p(x_1, ..., x_n) = 0 \end{cases}.$$

The Jacobian matrix of the *derivative* of $F(\boldsymbol{x})$ is represented by

$$F'(\boldsymbol{x}) = \begin{bmatrix} \dfrac{\partial f_1(x_1, ..., x_n)}{\partial x_1}, \cdots, \dfrac{\partial f_1(x_1, ..., x_n)}{\partial x_n} \\ \vdots \qquad\qquad \vdots \\ \dfrac{\partial f_p(x_1, ..., x_n)}{\partial x_1}, \cdots, \dfrac{\partial f_p(x_1, ..., x_n)}{\partial x_n} \end{bmatrix}.$$

The condition $F'(\boldsymbol{x})\ \delta\boldsymbol{x} = \mathbf{0}$ is written in matrix form as

$$\begin{bmatrix} \dfrac{\partial f_1(x_1,...,x_n)}{\partial x_1}, \ \dots\ , \dfrac{\partial f_1(x_1,...,x_n)}{\partial x_n} \\ \vdots \qquad\qquad\quad \vdots \\ \dfrac{\partial f_p(x_1,...,x_n)}{\partial x_1}, \ \dots\ , \dfrac{\partial f_p(x_1,...,x_n)}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \\ \vdots \\ \\ \delta x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the final column is made up of p lines.

The condition for the extremum of G can then be written in matrix form as

$$\begin{bmatrix} \dfrac{\partial G(x_1,...,x_n)}{\partial x_1}, \ \dots\ , \dfrac{\partial G(x_1,...,x_n)}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \\ \vdots \\ \\ \delta x_n \end{bmatrix} = 0.$$

Applying Theorem 1.1, there exists a Lagrange multiplier Λ, which is a linear mapping from $\mathbb{R}^p$ to $\mathbb{R}$ such that

$$G'(\boldsymbol{x}) = \Lambda\, F'(\boldsymbol{x}) \quad \text{with} \quad F(\boldsymbol{x}) = 0.$$

On the basis of $\mathbb{R}^{p\,\star}$, we write

$$\Lambda = [\lambda_1, ..., \lambda_p] \ \text{ or } \ \lambda_i \in \mathbb{R},\ i \in \{1, ..., p\}\,.$$

The following three properties are equivalent:

(A) $\forall\, \delta\boldsymbol{x},\ \delta\boldsymbol{x} \in \mathbb{R}^n,\ F'(\boldsymbol{x})\ \delta\boldsymbol{x} = \mathbf{0} \quad \Longrightarrow \quad G'(\boldsymbol{x})\delta\boldsymbol{x} = 0 \quad$ with $F(\boldsymbol{x}) = \mathbf{0}$;

(B) $G'(\boldsymbol{x}) = \Lambda\, F'(\boldsymbol{x})$ with $F(\boldsymbol{x}) = \mathbf{0}$;

(C) $\forall\, \delta\boldsymbol{x},\ \delta\boldsymbol{x} \in \mathbb{R}^n$ and $\forall\, \delta\Lambda,\ \delta\Lambda \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}),\ \delta\,[G(\boldsymbol{x}) - \Lambda\, F(\boldsymbol{x})] = 0.$

We have demonstrated that *(A)* is equivalent to *(B)*. Property *(C)* shows that finding a connected extremum is the same as finding a free extremum, but with respect to $n+p$ variables, the position of the extremum being made up of the elements $\boldsymbol{x}$ and Λ. The introduction of a Lagrange multiplier leads to the freeing of the constraint. This property results from Theorem 1.1, and we can write that $G(\boldsymbol{x}) - \Lambda\, F(\boldsymbol{x}) = b$, where b is the new function to stationarize.

1.3. The fundamental lemma of the calculus of variations

In the case where the unknown is an element of a functional space, we are truly dealing with the calculus of variations – this is an extension of differential calculus. Indeed, in the preceding sections, we have used the following lemma:

Given $\boldsymbol{A}$ belonging to $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$, for any $\boldsymbol{V}$ belonging to $\mathbb{R}^n$, $\boldsymbol{AV} = \boldsymbol{0}$ implies $\boldsymbol{A} = \boldsymbol{0}$.

It is possible to propose the following generalization:

Let us consider $\mathcal{E}$, the set of p-time differentiable real functions defined by

$$[t_0, t_1] \subset \mathbb{R} \longrightarrow \psi(t) \in \mathbb{R}^n.$$

If $p = 0$, functions are simply continuous. We may add $\psi(t_0) = \boldsymbol{0}$ and $\psi(t_1) = \boldsymbol{0}$. The set $\mathcal{E}$ has the structure of a real vector space. Let ϕ be a mapping

$$[t_0, t_1] \subset \mathbb{R} \longrightarrow \phi(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \equiv \mathbb{R}^{n\star},$$

then ϕ, defined from $\mathbb{R}$ to the dual of $\mathbb{R}^n$, is assumed to be continuous over $[t_0, t_1]$.

Let us write $\mathcal{G}(\psi) = \displaystyle\int_{t_0}^{t_1} \phi(t)\psi(t)dt$. This mapping defined over $\mathcal{E}$ is linear as for all scalars λ_1, λ_2 and for all functions ψ_1, ψ_2,

$$\mathcal{G}(\lambda_1\,\psi_1 + \lambda_2\,\psi_2) = \lambda_1\,\mathcal{G}(\psi_1) + \lambda_2\,\mathcal{G}(\psi_2).$$

LEMMA 1.1.– Let ϕ be a continuous mapping from $[t_0, t_1]$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ and ψ a continuous mapping from $[t_0, t_1]$ to $\mathbb{R}^n$; then,

$$\textit{For any } \psi, \quad \int_{t_0}^{t_1} \phi(t)\,\psi(t)dt = 0 \quad \textit{implies} \quad \phi = \boldsymbol{0}.$$

PROOF.– It is enough to demonstrate the lemma for $n = 1$. Let us assume through *reductio ad absurdum* that there exists $t_2 \in]t_0, t_1[$ such that $\phi(t_2) \neq 0$ (e.g. the $\phi(t_2)$ value is strictly positive). Since $\phi(t)$ is continuous, there exists $[t_2, t_3]$ included in $]t_0, t_1[$ such that for t belonging to $[t_2, t_3]$, $\phi(t) > 0$. By choosing $\psi\,(t) = [(t - t_2)(t_3 - t)]^{p+1}$ for $t \in [t_2, t_3]$ and $\psi\,(t) = 0$ for $t \notin [t_2, t_3]$, we would have

$$\int_{t_0}^{t_1} \phi(t)\psi(t)dt = \int_{t_2}^{t_3} \phi(t)\,[(t - t_2)(t_3 - t)]^{p+1}\,dt > 0,$$

which leads to a contradiction. This lemma can easily be extended when t_2 takes the value of one of its limits t_0 or t_1 and can be applied in the case where $\psi(t_0) = \psi(t_1) = 0$ (as was the case in the proof). ■

LEMMA 1.2.– ***du Bois-Raymond's lemma***. ϕ being a continuous mapping from $[t_0, t_1]$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ and ψ being an application with a continuous derivative from $[t_0, t_1]$ to $\mathbb{R}^n$ such that $\psi(t_0) = \psi(t_1) = \mathbf{0}$; then,

$$For\ any\ \psi, \quad \int_{t_0}^{t_1} \phi(t)\,\psi'(t)\,dt = 0 \quad implies \quad \phi = \boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$.

Then, ψ belongs to $D_1[t_0, t_1]$.

PROOF.– It is enough to demonstrate the lemma for $n = 1$; this can easily be extended to $n > 1$. Let the real c defined by $\displaystyle\int_{t_0}^{t_1} (\phi(t) - c)\ dt = 0$. Let us write $\psi(t) = \displaystyle\int_{t_0}^{t} (\phi(u) - c)\ du$. We have $\psi(t) \in D_1[t_0, t_1]$. According to the hypotheses,

$$\int_{t_0}^{t_1} (\phi(t) - c)\,\psi'(t)\,dt = \int_{t_0}^{t_1} \phi(t)\,\psi'(t)\,dt - c\,(\psi(t_1) - \psi(t_0)) = 0.$$

However, $\psi'(t) = \phi(t) - c$; hence, $\displaystyle\int_{t_0}^{t_1} (\phi(t) - c)^2\ dt = 0$, which implies $\phi(t) = c$. ■

Let us note that we have written $\mathcal{G}(\psi) = \displaystyle\int_{t_0}^{t_1} \phi(t)\,\psi'(t)\,dt$; $\mathcal{G}$ is a linear mapping from $\mathcal{E}$ to $\mathbb{R}$; consequently, $\mathcal{G}$ is a linear functional of $\mathcal{E}$ or an element in the dual vector space $\mathcal{E}^*$. The parallel with section 1.2 is complete.

1.4. Extremum of a free functional

Let G be a continuously differentiable mapping from $\Omega \times [t_0, t_1]$ to $\mathbb{R}$, with $\Omega \subset \mathbb{R}^n$. This mapping is denoted by

$$(\boldsymbol{Q}, t) \in \Omega \times [t_0, t_1] \longrightarrow G(\boldsymbol{Q}, t) \in \mathbb{R}$$

and we say that G is a *generating function*. Let ϕ be a continuous mapping from $[t_0, t_1]$ to $\mathbb{R}^n$. We posit

$$a = \int_{t_0}^{t_1} G(\phi(t), t)dt,$$

and we write $a = \mathcal{G}(\phi)$; thus, $\mathcal{G}$ is a functional of the ϕ-functions and belongs to $\mathcal{A}(\mathcal{A}(\mathbb{R}, \mathbb{R}^n), \mathbb{R})$.

Let ψ be another continuous mapping from $[t_0, t_1]$ to $\mathbb{R}^n$. As the scalar x is real, $\phi + x\psi$ is a continuous application from $[t_0, t_1]$ to $\mathbb{R}^n$ and

$$\mathcal{G}(\phi + x\psi) = \int_{t_0}^{t_1} G\big(\phi(t) + x\psi(t), t\big)dt.$$

For the given ϕ and ψ, we write $g(x) = \mathcal{G}(\phi + x\psi)$. Thus, the mapping g is a linear mapping from $\mathbb{R}$ to $\mathbb{R}$. For the given ϕ, ψ and t, we write $f(x) = G\big(\phi(t) + x\,\psi(t), t\big)$. The function f can be derived at $x = 0$, i.e.

$$f(x) = f(0) + xf'(0) + x\,\varepsilon(x) \quad \text{where} \quad \lim_{x\to 0} \varepsilon(x) = 0.$$

Consequently, as $\dfrac{\partial G\big(\phi(t), t\big)}{\partial \boldsymbol{Q}}$ is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$,

$$\begin{aligned} f(x) = G\big(\phi(t) + x\psi(t), t\big) &= G\big(\phi(t), t\big) + x\,\frac{\partial G}{\partial \boldsymbol{Q}}\big(\phi(t), t\big)\ \psi(t) \\ &+x\ \|\psi(t)\|\,\varepsilon\big(x\psi(t)\big), \\ \text{where} \quad \lim_{x\to 0} \varepsilon\big(x\psi(t)\big) &= 0, \end{aligned}$$

and, through integration

$$\begin{aligned} \int_{t_0}^{t_1} G\big(\phi(t) + x\psi(t), t\big)dt &= \int_{t_0}^{t_1} G\big(\phi(t), t\big)dt \\ &\quad +x\int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}\big(\phi(t), t\big)\ \psi(t)dt + x\varepsilon_1(x) \\ \text{where} \quad \varepsilon_1(x) &= \int_{t_0}^{t_1} \|\psi(t)\|\,\varepsilon\big(x\psi(t)\big)dt \ \text{ with } \lim_{x\to 0} \varepsilon_1(x) = 0. \end{aligned}$$

$$\text{Thus ,} \qquad \mathcal{G}(\phi + x\psi) = \mathcal{G}(\phi) + x\int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}\big(\phi(t), t\big)\ \psi(t)dt + x\varepsilon_1(x),$$

COROLLARY 1.1.–

$$g'(0) = \int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), t)\,\psi(t)\,dt,$$

which corresponds to the derivative of g at 0.

DEFINITION 1.2.– $a = \int_{t_0}^{t_1} G\big(\phi(t), t\big)dt$ is an extremum for the continuous function ϕ from $[t_0, t_1]$ to $\mathbb{R}^n$ if and only if for any ψ continuous mapping from $[t_0, t_1]$ to $\mathbb{R}^n$,

$$g'(0) \equiv \int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}\big(\phi(t), t\big)\ \psi(t)dt = 0.$$

We write $\delta a = g'(0)$, which is called *the variation of* a *relative to the continuous functions* ϕ *from* $[t_0, t_1]$ *to* $\mathbb{R}^n$. When there is no ambiguity, we write

$$a = \int_{t_0}^{t_1} G(\boldsymbol{Q}, t)\, dt,$$

and, analogous with the previous sections, $\psi(t)$ is denoted by $\delta\boldsymbol{Q}(t)$ or, more simply, $\delta\boldsymbol{Q}$. We write

$$g'(0) = \delta a = \int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}(Q, t)\ \delta\boldsymbol{Q}\, dt.$$

From Lemma 1.2 and Corollary 1.1, it follows that

COROLLARY 1.2.– $a = \int_{t_0}^{t_1} G\big(\boldsymbol{Q}, t\big)dt$ is an extremum for the continuous function $\boldsymbol{Q}$ from $[t_0, t_1]$ to $\mathbb{R}^n$ if $\dfrac{\partial G(\boldsymbol{Q}, t)}{\partial \boldsymbol{Q}} = \boldsymbol{0}$.

When there is no ambiguity, we write $\dfrac{\partial G}{\partial Q}$ for $\dfrac{\partial G(Q, t)}{\partial Q}$.

1.5. Extremum for a constrained functional

The search for the function $t \in [t_0, t_1] \longrightarrow \phi(t) \in \mathbb{R}^n$ may satisfy certain relationships called constraints. We come across two types of constraints.

1.5.1. *First type: integral constraint*

DEFINITION 1.3.– An integral constraint is a relationship in the form

$$\int_{t_0}^{t_1} F(\phi(t), t)\, dt = b, \qquad [1.1]$$

where b is a given real and F is a continuously differentiable mapping from an open set Ω of $\mathbb{R}^n \times [t_0, t_1]$ to $\mathbb{R}$.

We can generalize the definition for a mapping from $\mathbb{R}^n \times [t_0, t_1]$ to $\mathbb{R}^p$. The generalization corresponds to p integral constraints. Finding the mappings ϕ which will make $\displaystyle\int_{t_0}^{t_1} G(\phi(t), t)dt$ extremum and which are subject to condition [1.1] results in the shortened form:

$$\forall\, \psi \text{ continue, } \psi \in \mathcal{A}(\mathbb{R}, \mathbb{R}^p),$$
$$\delta b \equiv \int_{t_0}^{t_1} \frac{\partial F}{\partial \boldsymbol{Q}}(\phi(t), t)\, \psi(t)dt = 0 \implies \delta a \equiv \int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), t)\, \psi(t)dt = 0.$$

We can simply write: $\delta b = 0 \implies \delta a = 0$.

THEOREM 1.2.– With δb and δa being two linear functionals defined over the continuous mappings ϕ from $\mathbb{R}$ to $\mathbb{R}^n$, with values in $\mathbb{R}$,

$$\delta b = 0 \quad \Longrightarrow \quad \delta a = 0$$

is equivalent to there existing a Lagrange multiplier, i.e. a linear mapping from $\mathbb{R}$ to $\mathbb{R}$ such that $\delta a = \lambda\, \delta b$. We can write

$$\int_{t_0}^{t_1} \frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), t)\, \psi(t)dt = \lambda \int_{t_0}^{t_1} \frac{\partial F}{\partial \boldsymbol{Q}}(\phi(t), t)\, \psi(t)dt.$$

Thus, λ is a real constant.

We are brought back to the search for the free extremum of

$$a - \lambda b = c \quad \text{with} \quad c = \int_{t_0}^{t_1} [\, G(\phi(t), t) - \lambda\, F(\phi(t), t)\,]\, dt.$$

Lemma 1.1 of the calculus of variations implies the following theorem.

THEOREM 1.3.– The continuous functions ϕ from $[t_0, t_1]$ to $\mathbb{R}^n$, satisfying relationship [1.1] and which makes "integral a" extremum, simultaneously satisfy the two relations:

$$\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), t) = \lambda \frac{\partial F}{\partial \boldsymbol{Q}}(\phi(t), t) \quad \text{and} \quad \int_{t_0}^{t_1} F(\phi(t), t)dt = b.$$

We also use the result in simplified form as

$$\frac{\partial G}{\partial \boldsymbol{Q}}(\boldsymbol{Q}(t), t) = \lambda \frac{\partial F}{\partial \boldsymbol{Q}}(\boldsymbol{Q}(t), t) \quad \text{and} \quad \int_{t_0}^{t_1} F(\boldsymbol{Q}(t), t)dt = b.$$

These properties generalize, for functionals with real values, the results obtained in section 1.4.

PROOF.– We demonstrate the property in the case of a mapping ϕ from $\mathbb{R}$ to $\mathbb{R}$. We can easily generalize the demonstration to the mappings with values in $\mathbb{R}^n$. Let us write

$$a = \int_{t_0}^{t_1} g(x(t), t)dt \text{ and } b = \int_{t_0}^{t_1} f(x(t), t)dt,$$

where f and g are two continuously differentiable mappings from $\mathbb{R}^2$ to $\mathbb{R}$ and $x(t)$ denotes a continuous mapping from $\mathbb{R}$ to $\mathbb{R}$. Thus,

$$\delta a = \int_{t_0}^{t_1} \frac{\partial g}{\partial x}(x(t), t)\delta x\, dt \text{ and } \delta b = \int_{t_0}^{t_1} \frac{\partial f}{\partial x}(x(t), t)\delta x\, dt.$$

Let us write

$$\mathcal{J}(\delta x) = \int_{t_0}^{t_1} \frac{\partial g}{\partial x}(x(t), t)\, \delta x\, dt \quad \text{and} \quad \mathcal{K}(\delta x) = \int_{t_0}^{t_1} \frac{\partial f}{\partial x}(x(t), t)\, \delta x\, dt.$$

The terms $\mathcal{J}$ and $\mathcal{K}$ are two linear mappings defined over δx mappings from $\mathbb{R}$ to $\mathbb{R}$. Thus,

$$\forall\, \delta x,\ \delta x \in C^0(\mathbb{R}, \mathbb{R}), \quad \mathcal{K}(\delta x) = 0 \quad \Longrightarrow \quad \mathcal{J}(\delta x) = 0.$$

Let δx_1 be a given continuous mapping from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathcal{K}(\delta x_1) \neq 0$ and δx is any other continuous mapping from $\mathbb{R}$ to $\mathbb{R}$. Let us write $\delta x_2 = \delta x - \mu\, \delta x_1$, where $\mu \in \mathbb{R}$ and such that $\mathcal{K}(\delta x_2) = 0$. This implies that $\mathcal{K}(\delta x - \mu\, \delta x_1) = 0$ or $\mu = \mathcal{K}(\delta x)/\mathcal{K}(\delta x_1) \in \mathbb{R}$. According to the hypothesis, we obtain $\mathcal{K}(\delta x_2) = 0 \Longrightarrow \mathcal{J}(\delta x_2) = 0$. From this result, it can be derived that

$$\begin{aligned} \mathcal{J}(\delta x_2) = \mathcal{J}(\delta x) - \mu\, \mathcal{J}(\delta x_1) = 0 \quad &\Longrightarrow \quad \mathcal{J}(\delta x_2) \\ = \mathcal{J}(\delta x) - \mathcal{K}(\delta x)\, &\frac{\mathcal{J}(\delta x_1)}{\mathcal{K}(\delta x_1)} = 0. \end{aligned}$$

Let us write $\lambda = \mathcal{J}(\delta x_1)/\mathcal{K}(\delta x_1)$; λ is a scalar that is given by δx_1. Hence, $\mathcal{J}(\delta x) = \lambda\, \mathcal{K}(\delta x)$. ■

COROLLARY 1.3.– The three propositions are equivalent:

$$(A)\ \forall\ \delta x, \quad \mathcal{K}(\delta x) = 0 \quad \Longrightarrow \quad \mathcal{J}(\delta x) = 0;$$

$$(B)\ \exists\ \lambda \in \mathbb{R} \text{ such that } \forall\ \delta x,\ \delta x \in C^0(\mathbb{R}, \mathbb{R}),\ \mathcal{J}(\delta x) - \lambda\, \mathcal{K}(\delta x)\ = 0;$$

$$(C)\ \exists\ \lambda \in \mathbb{R},\ \text{a constant scalar, called the Lagrange multiplier, such that}$$
$$\mathcal{J} = \lambda\, \mathcal{K}.$$

We also write $\delta a - \lambda\, \delta b = 0$.

1.5.2. *Second type: distributed constraint*

Let F be a mapping from $\Omega \times [t_0, t_1]$ to $\mathbb{R}$, where Ω is an open set of $\mathbb{R}^n$ and $[t_0, t_1]$ a segment of $\mathbb{R}$. We wish to find the extrema of $a = \displaystyle\int_{t_0}^{t_1} G(\boldsymbol{Q}(t), t)dt$ such that, for any value t of $[t_0, t_1]$, we have $F(\boldsymbol{Q}(t), t) = 0$. The constraint $F(\boldsymbol{Q}(t), t) = 0$ is called a distributed constraint.

We demonstrate that we are led to introduce a multiplier denoted by Λ, a function defined for each value of t belonging to $[t_0, t_1]$, i.e. a Λ-multiplier function of t.

THEOREM 1.4.– We have the equivalence of the three propositions:

(A) there exists a continuous mapping $\boldsymbol{Q}$ from $[t_0, t_1]$ to Ω satisfying $F(\boldsymbol{Q}(t), t) = 0$ such that for any continuous application $\delta\boldsymbol{Q}$ from $[t_0, t_1]$ to $\mathbb{R}^n$ and for any t of $[t_0, t_1]$,

$$\frac{\partial F}{\partial \boldsymbol{Q}}(\boldsymbol{Q}(t), t)\ \delta\boldsymbol{Q} = 0 \quad \Longrightarrow \quad \delta \int_{t_0}^{t_1} G(\boldsymbol{Q}(t), t)dt = 0;$$

(B) there exists a continuous mapping $\boldsymbol{Q}$ from $[t_0, t_1]$ to Ω and there exists a continuous mapping Λ from $[t_0, t_1]$ to $\mathbb{R}$ such that for any continuous mapping $\delta\boldsymbol{Q}$ from $[t_0, t_1]$ to $\mathbb{R}^n$ and any continuous mapping $\delta\Lambda$ from $[t_0, t_1]$ to $\mathbb{R}$,

$$\delta \int_{t_0}^{t_1} \big(G(\boldsymbol{Q}(t), t) - \Lambda(t)\, F(\boldsymbol{Q}(t), t)\big)dt = 0;$$

(C) there exists a continuous mapping $\boldsymbol{Q}$ from $[t_0, t_1]$ to Ω and there exists a continuous mapping Λ from $[t_0, t_1]$ to $\mathbb{R}$ such that

$$\frac{\partial G}{\partial \boldsymbol{Q}}(\boldsymbol{Q}(t), t) - \Lambda(t)\ \frac{\partial F}{\partial Q}(\boldsymbol{Q}(t), t) = 0 \ \text{ and } \ F(\boldsymbol{Q}(t), t) = 0.$$

PROOF.– Let us demonstrate the equivalence of *(A)*, *(B)* and *(C)*.

(B) is equivalent to *(C)*. This result leads to the writing of the extrema of

$$\int_{t_0}^{t_1} \left(G(Q(t),t) - \Lambda(t)\, F(Q(t),t)\right) dt.$$

(C) implies *(A)*. Indeed, $\delta \int_{t_0}^{t_1} G(\boldsymbol{Q}(t),t)\, dt = \int_{t_0}^{t_1} \dfrac{\partial G(\boldsymbol{Q}(t),t)}{\partial \boldsymbol{Q}}\, \delta\boldsymbol{Q}(t)\, dt.$

Let $\delta\boldsymbol{Q}(t)$ be such that $\dfrac{\partial F(\boldsymbol{Q}(t),t)}{\partial Q}\, \delta\boldsymbol{Q}(t) = 0$, then $\dfrac{\partial G(\boldsymbol{Q}(t),t)}{\partial \boldsymbol{Q}}\, \delta\boldsymbol{Q}(t) = 0$ and $\delta \int_{t_0}^{t_1} G(\boldsymbol{Q}(t),t)dt = 0$.

(A) implies *(C)*. In order to simplify the demonstration, it is assumed that the mapping F has real values; the demonstration can easily be extended for $n > 1$. Given $\boldsymbol{Q}(t)$ satisfying $F(\boldsymbol{Q}(t),t) = 0$ and making $\int_{t_0}^{t_1} G(\boldsymbol{Q}(t),t)dt$ extremum, the mapping $\boldsymbol{Q}(t)$ also makes the integral,

$$b = \int_{t_0}^{t_1} \left(G(\boldsymbol{Q}(t),t) - \Lambda(t)\, F(\boldsymbol{Q}(t),t)\right) dt,$$

extremum. This is true for any mapping Λ from $[t_0, t_1]$ to $\mathbb{R}$.

Let us calculate δb for any variation of $\delta\boldsymbol{Q}(t) = [\delta q_1(t), \ldots, \delta q_n(t)]^\star$

$$\delta b = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial G}{\partial q_1} - \Lambda \frac{\partial F}{\partial q_1}\right] \delta q_1 + \left[\frac{\partial G}{\partial q_2} - \Lambda \frac{\partial F}{\partial q_2}\right] \delta q_2 + \ldots \right.$$
$$\left. + \left[\frac{\partial G}{\partial q_n} - \Lambda \frac{\partial F}{\partial q_n}\right] \delta q_n \right\} dt.$$

We can write that $\delta b = 0$ for any $\delta q_1, \delta q_2, \ldots, \delta q_n$ satisfying

$$\frac{\partial F}{\partial q_1}\, \delta q_1 + \ldots + \frac{\partial F}{\partial q_n}\, \delta q_n = 0. \qquad [1.2]$$

Let us assume that $\dfrac{\partial F}{\partial q_1}$ does not cancel between t_0 and t_1. The function Λ, which was arbitrary until now, is given by the relationship

$$\frac{\partial G}{\partial q_1} - \Lambda\, \frac{\partial F}{\partial q_1} = 0, \qquad [1.3]$$

and we get,

$$\delta b = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial G}{\partial q_2} - \Lambda \frac{\partial F}{\partial q_2} \right] \delta q_2 + \ldots + \left[\frac{\partial G}{\partial q_n} - \Lambda \frac{\partial F}{\partial q_n} \right] \delta q_n \right\} dt$$

which must be zero for any value of $\delta q_2, \ldots, \delta q_n$, which are no longer constrained by relation [1.2] since δq_1 is derived from this relationship. The fundamental lemma for the calculus of variations gives: for any i belonging to $\{2, \ldots, n\}$,

$$\frac{\partial G}{\partial q_i} - \Lambda \frac{\partial F}{\partial q_i} = 0 \qquad [1.4]$$

and relationships [1.3, 1.4] imply *(C)* ■

Let us note that if, for a value t_2 in the interval $[t_0, t_1]$, $\dfrac{\partial F}{\partial q_1} = 0$, we choose Λ in a neighborhood of t_2, satisfying the condition $\dfrac{\partial G}{\partial q_p} - \Lambda \dfrac{\partial F}{\partial q_p} = 0$, where at t_2, $\dfrac{\partial F}{\partial q_p} \neq 0$.

The only case that would prevent the existence of Λ for a particular value t_2 in the interval $[t_0, t_1]$ would be when $\dfrac{\partial F(\boldsymbol{Q}(t_2), t_2)}{\partial \boldsymbol{Q}} = 0$. This case corresponds to a singular point of F. We then say that the constraint is singular at t_2.

In the form of meta-language, the results that represent the equivalent conditions for a distributed constrained condition are summarized as:

(A) $\quad \exists\, \boldsymbol{Q} \in C^0([t_0, t_1], \mathbb{R}^n)$ with $F(\boldsymbol{Q}, t) = 0$ such that

$$\forall\, \delta\boldsymbol{Q},\ \frac{\partial F}{\partial \boldsymbol{Q}}\, \delta\boldsymbol{Q} = 0 \implies \delta \int_{t_0}^{t_1} G(\boldsymbol{Q}(t), t) dt = 0,$$

(B) $\quad \exists\, \boldsymbol{Q} \in C^0([t_0, t_1], \mathbb{R}^n),\ \exists\, \Lambda \in C^0([t_0, t_1], \mathbb{R})$ such that

$$\forall\, \delta\boldsymbol{Q},\ \forall\, \delta\Lambda,\ \delta \int_{t_0}^{t_1} (G(\boldsymbol{Q}, t) - \Lambda\, F(\boldsymbol{Q}, t))\, dt = 0,$$

(C) $\quad \exists\, \boldsymbol{Q} \in C^0([t_0, t_1], \mathbb{R}^n),\ \exists\, \Lambda \in C^0([t_0, t_1], \mathbb{R})$ such that

$$\frac{\partial G}{\partial \boldsymbol{Q}} - \Lambda \frac{\partial F}{\partial \boldsymbol{Q}} = 0 \text{ and } F(\boldsymbol{Q}(t), t) = 0.$$

1.6. More general problem of the calculus of variations

1.6.1. *First extension of the earlier results*

Let G be a continuously differentiable mapping from $\Omega \times \ldots \times \Omega \times [t_0, t_1]$ to $\mathbb{R}$ (where Ω, an open set of $\mathbb{R}^n$, is repeated q times) and ϕ is a mapping from $[t_0, t_1]$ to Ω, p times derivable ($p > q$; if $p = q$, the pth derivative is continuous). The set of functions ϕ is called $(\mathcal{F})$. Let us write

$$a = \int_{t_0}^{t_1} G\big(\phi(t), \phi'(t), \ldots, \phi^{(q)}(t), t\big) dt.$$

We wish to find, among the $(\mathcal{F})$ functions, the functions which make a an extremum. The mappings ϕ of $(\mathcal{F})$ can be subject to additional constraints of the type that we studied in the previous section:

(*L1*) Integral constraint of the form

$$\int_{t_0}^{t_1} F\big(\phi(t), \phi'(t), \ldots, \phi^{(r)}(t), t\big)\, dt = b \text{ where } r \leq p.$$

Such a constraint corresponds to a constant Lagrange multiplier; if the constraint is scalar (i.e. if b is scalar), then the Lagrange multiplier is a given real λ.

(*L2*) Distributed constraint of the form

$$H\big(\phi(t), \phi'(t), \ldots, \phi^{(r)}(t), t\big) = 0, \text{ where } r \leq p.$$

The functions F and H have the same conditions for regularity as G. If H has scalar values, we associate the constraint (*L2*) with a Lagrange multiplier λ, which is a mapping from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, $\phi(t)$ can be subject to additional conditions of the type

$$\phi(t_0) = \phi(t_1) = \mathbf{0},$$

and even to conditions related to the derivatives of $\phi(t)$ at t_0 and t_1. We do not study the most general case, but the reasoning used will remain the same.

1.6.2. *Important example*

Based on the hypotheses in section 1.6.1, we propose studying the variations of

$$G(\phi) = \int_{t_0}^{t_1} G(\phi(t), \phi'(t), t)\, dt, \text{ which can be written as } \quad a = \int_{t_0}^{t_1} G(\boldsymbol{Q}, \boldsymbol{Q}', t)\, dt.$$

Consider the mapping $\phi(t) + x\ \psi(t)$, where ϕ and ψ are two derivable mappings of the real variable with real values. Thus,

$$G(\phi + x\ \psi) = \int_{t_0}^{t_1} G(\phi(t) + x\ \psi(t), \phi'(t) + x\ \psi'(t), t)\, dt.$$

As in section 1.5, we denote for the given ϕ and ψ, $f(x) = G(\phi + x\ \psi)$ and $\delta a = f'(0)$. Then,

$$\delta a = \int_{t_0}^{t_1} \left[\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), \phi'(t), t)\, \psi(t) + \frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\, \psi'(t) \right] dt.$$

Since

$$\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\, \psi'(t) = \left[\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\, \psi(t) \right]' - \frac{d}{dt} \left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t) \right) \psi(t),$$

we can derive

$$\delta a = \left[\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\, \psi(t) \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), \phi'(t), t) - \frac{d}{dt} \left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t) \right) \right] \psi(t) dt. \qquad [1.5]$$

– In the particular case where $\psi(t_0) = \psi(t_1) = \mathbf{0}$, corresponding to null function values at the extremities of the segment $[t_0, t_1]$, we can derive

$$\delta a = \int_{t_0}^{t_1} \left[\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), \phi'(t), t) \ - \frac{d}{dt} \left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t) \right) \right] \psi(t) dt$$

and the fundamental lemma of the calculus of variations implies

$$\frac{d}{dt} \left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t) \right) - \frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), \phi'(t), t) \ = 0. \qquad [1.6]$$

– In the most general case where $\psi(t_0)$ and $\psi(t_1)$ are arbitrary, we can choose for them to be zero at t_0 and t_1 and we arrive again at equation [1.6]. Consequently, by writing this result in equation [1.5], for any $\psi(t_0)$ and $\psi(t_1)$,

$$\delta a = \left[\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\ \psi(t) \right]_{t_0}^{t_1} = 0$$

and we obtain

$$\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t_0), \phi'(t_0), t_0) = \mathbf{0} \quad \text{and} \quad \frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t_1), \phi'(t_1), t_1) = \mathbf{0},$$

corresponding to the fact that δa is a linear functional of both $\psi(t)$ and $\psi(t_0)$, $\psi(t_1)$.

We simply write

$$a = \int_{t_0}^{t_1} G(\boldsymbol{Q}, \boldsymbol{Q}', t) dt,$$

hence,

$$\delta a = \int_{t_0}^{t_1} \left(\frac{\partial G}{\partial \boldsymbol{Q}} \delta \boldsymbol{Q} + \frac{\partial G}{\partial \boldsymbol{Q}'} \delta \boldsymbol{Q}' \right) dt \text{ with } \delta \boldsymbol{Q}' = (\delta \boldsymbol{Q})',$$

which implies

$$\delta a = \left[\frac{\partial G}{\partial \boldsymbol{Q}'} \delta \boldsymbol{Q} \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\frac{\partial G}{\partial \boldsymbol{Q}} - \frac{d}{dt} \left(\frac{\partial G}{\partial \boldsymbol{Q}'} \right) \right] \delta \boldsymbol{Q} \, dt.$$

Thus, δa is the sum of a linear functional of $\delta \boldsymbol{Q}$ and a linear function of $\delta \boldsymbol{Q}_0$ and $\delta \boldsymbol{Q}_1$. We can derive the equations known as *the Euler equations*:

$$\frac{d}{dt} \left(\frac{\partial G}{\partial Q'} \right) - \frac{\partial G}{\partial Q} = 0 \quad \text{or} \quad \frac{d}{dt} \left(\frac{\partial G}{\partial q_i'} \right) - \frac{\partial G}{\partial q_i} = 0 \text{ for } i \in \{1, \ldots, n\}. \qquad [1.7]$$

The method used can be summarized in the following way: with $\boldsymbol{Q}$ being the solution to the variational problem, $\delta \boldsymbol{Q}$ is a variation of $\boldsymbol{Q}$ and this results in the variation δa. Through integration by parts, we apply the fundamental lemma for the calculus of variations, which gives the equation satisfied by $\boldsymbol{Q}$.

1.6.3. *First application: the Hamilton principle*

Consider a conservative mechanical system whose position is determined with respect to a reference Galilean system using n *independent parameters*. The system is said to have n degrees of freedom. The constraints that limit the position of the system are assumed to be perfect. The movements are represented by the C^2 class $\phi(t)$ mappings from $t \in [t_0, t_1]$ to $\mathbb{R}^n$.

PRINCIPLE 1.1.– ***The Hamilton principle***:

The motion of the mechanical system between instants t_0 and t_1 associated with the initial position $\boldsymbol{Q}_0 = \phi(t_0)$ and the final position $\boldsymbol{Q}_1 = \phi(t_1)$ makes

$$a = \int_{t_0}^{t_1} G(\phi(t), \phi'(t), t)dt \quad extremum.$$

In the Hamilton principle, G is called the *Lagrangian* and its value is the difference between the system's kinetic energy T and potential energy W: $G \equiv T - W$. The integral a is called the *Hamiltonian action* of the system.

Observations

– We have assumed that the possible positions of the system are represented using n independent parameters. It is the explicit solution of the constraint equations that allows us to obtain this result.

– The constraints are perfect; the system is said to be conservative or without dissipation.

– The equilibrium positions of the system are specific motions of the system: they are the motions for which the potential energy is extremum. This is the particular case where the kinetic energy, which is positive, is zero.

– The ϕ functions do not have the structure of a vector space. Indeed, if ϕ_1 and ϕ_2 belong to $(\mathcal{F})$, where $(\mathcal{F})$ is the set of C^2 functions such that $\phi(t_0) = \boldsymbol{Q}_0 \neq \boldsymbol{0}$ and $\phi(t_1) = \boldsymbol{Q}_1$, then $\phi_1(t_0) + \phi_2(t_0) = 2\boldsymbol{Q}_0$. We would then have to carry out the following modification: the ψ functions, which can also be denoted by $\delta\boldsymbol{Q}$, are the difference between the two $(\mathcal{F})$ functions. The set that is defined is denoted by $(\mathcal{E})$. This set is the vector space of the C^2 functions that are zero at t_0 and at t_1.

For any real value x, for all the ϕ functions of $(\mathcal{F})$ and ψ functions of $(\mathcal{E})$, $\phi + x\,\psi$ belongs to $(\mathcal{F})$, since

$$\phi(t_0) + x\,\psi(t_0) = \phi(t_0) = \boldsymbol{Q}_0 \text{ and } \phi(t_1) + x\,\psi(t_1) = \phi(t_1) = \boldsymbol{Q}_1\,,$$

therefore,

$$\delta a = \frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t_1), \phi'(t_1), t_1)\,\psi(t_1) - \frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t_0), \phi'(t_0), t_0)\,\psi(t_0)$$
$$+ \int_{t_0}^{t_1} \left[\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t), \phi'(t), t) - \frac{d}{dt}\left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t), \phi'(t), t)\right)\right] \psi(t)\,dt,$$

and the Hamilton principle makes it possible to write

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \boldsymbol{Q}'}(\phi(t),\phi'(t),t)\right)-\frac{\partial G}{\partial \boldsymbol{Q}}(\phi(t),\phi'(t),t)=0. \qquad [1.8]$$

We obtain the *Euler equation of motion*. If $T = T(\phi(t),\phi'(t),t)$ and $W = W(\phi(t),t)$, equations [1.8] can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial Q'}\right)-\frac{\partial T}{\partial \boldsymbol{Q}}=-\frac{\partial W}{\partial \boldsymbol{Q}} \qquad [1.9]$$

which are called the *Lagrange equations.*

Observations

– $\dfrac{\partial G}{\partial Q}-\dfrac{d}{dt}\left(\dfrac{\partial G}{\partial Q'}\right)$ is the value at t of a linear form of $\mathbb{R}^{n\star}$ represented by a row matrix (1 row and n columns)

$$\left[\frac{\partial G}{\partial q_1}-\frac{d}{dt}\left(\frac{\partial G}{\partial q_1'}\right),\ldots,\frac{\partial G}{\partial q_n}-\frac{d}{dt}\left(\frac{\partial G}{\partial q_n'}\right)\right]$$

– The Lagrange equations are equivalent to n scalar equations

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial q_1'}\right)-\dfrac{\partial T}{\partial q_1}=-\dfrac{\partial W}{\partial q_1},\\ \vdots\\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial q_n'}\right)-\dfrac{\partial T}{\partial q_n}=-\dfrac{\partial W}{\partial q_n}.\end{cases}$$

Following Newtonian notations, if t is time, we then write $q_i' = \dot{q}_i$.

1.6.4. *Second application: geodesics of surfaces*

Consider an affine three-dimensional Euclidean space denoted by $\mathcal{E}^3$. A point M in this space is represented, with respect to a right-handed orthonormal system $\boldsymbol{O\,ijk}$, by the system with three coordinates (x,y,z). We also denote the bipoint $\boldsymbol{OM}$ as,

$$\boldsymbol{M}=\begin{bmatrix}x\\y\\z\end{bmatrix}.$$

The equation for a surface (S) takes the form $F(\boldsymbol{M}) \equiv F(x, y, z) = 0$, where F is a differentiable mapping from $\mathbb{R}^3$ to $\mathbb{R}$.

We wish to determine the curves of (S) with fixed extremities A and B, whose length is an extremum. We assume that the curves are represented by differentiable mappings from $\mathbb{R}$ to $\mathcal{E}^3$. One of these curves, which we call (C), is a geodesic of the surface joining $\boldsymbol{A}$ with $\boldsymbol{B}$. This geodesic will be represented by the mapping

$$u \in [u_0, u_1] \longrightarrow \boldsymbol{M} = f(u) \in \mathcal{E}^3 \text{ with } F(f(u)) = 0.$$

Let s denote a curvilinear abscissa of (C), then $d\boldsymbol{M} = \boldsymbol{T}\, ds$, where $\boldsymbol{T}$ denotes the unit vector tangent to (C)[1]. Thus, $(ds)^2 = (d\boldsymbol{M})^2$; $d\boldsymbol{M} = f'(u)\, du$, which we also write as $\boldsymbol{M}'_u\, du$. We obtain $(d\boldsymbol{M})^2 = \boldsymbol{M}'^{\star}_u \boldsymbol{M}'_u (du)^2 = \left(x'^2_u + y'^2_u + z'^2_u\right)\, (du)^2$, where $\boldsymbol{M}'_u$ is the vector of components x'_u, y'_u, z'_u; and let $\boldsymbol{M}'^{\star}_u$ be the row matrix representing $[x'_u, y'_u, z'_u]$.

Let us write $G(\boldsymbol{M}'_u) = \boldsymbol{M}'^{\star}_u \boldsymbol{M}'_u$; for an appropriate choice of orientation for (C), $ds = \sqrt{G(\boldsymbol{M}'_u)}\, du$. This leads to find the curves, (C), which make

$$L = \int_{u_0}^{u_1} \sqrt{G(\boldsymbol{M}'_u)} du$$

extremum, where $\boldsymbol{M}(u_0) = \boldsymbol{A}, \boldsymbol{M}(u_1) = \boldsymbol{B}$ and $F(\boldsymbol{M}(u)) = 0$.

According to section 1.5.2, associated with the existence of a distributed multiplier, the problem is equivalent to the following:

There exists a C^1-mapping $u \in [u_0, u_1] \longrightarrow \boldsymbol{M} = f(u) \in E^3$ with $f(u_0) = \boldsymbol{A}$ and $f(u_1) = \boldsymbol{B}$ and a C^0 mapping $u \in [u_0, u_1] \longrightarrow \lambda(u)$ such that

$$a = \int_{u_0}^{u_1} \left[\sqrt{G(\boldsymbol{M}'_u)} - \lambda(u)\, F(\boldsymbol{M}(u))\right]\, du$$

is extremum for $\boldsymbol{M} = f(u)$ and $\lambda(u)$. Furthermore,

$$d\sqrt{G(\boldsymbol{M}'_u)} = \frac{dG(\boldsymbol{M}'_u)}{2\sqrt{G(\boldsymbol{M}'_u)}} \quad \text{and} \quad dG(\boldsymbol{M}'_u) = d(\boldsymbol{M}'^{\star}_u \boldsymbol{M}'_u) = 2\boldsymbol{M}'^{\star}_u d\boldsymbol{M}'_u\,,$$

which derives

$$d\sqrt{G(\boldsymbol{M}'_u)} = \frac{\boldsymbol{M}'^{\star}_u d\boldsymbol{M}'_u}{\sqrt{G(\boldsymbol{M}'_u)}},$$

1 It is recalled that we write vectors without an arrow but in bold italic.

and implies

$$\frac{\partial\sqrt{G(\boldsymbol{M}'_u)}}{\partial\boldsymbol{M}'_u} = \frac{\boldsymbol{M}'^{\star}_u}{\sqrt{G(\boldsymbol{M}'_u)}} = \frac{\left(\dfrac{d\boldsymbol{M}}{du}\right)^{\star}}{\dfrac{ds}{du}} = \frac{d\boldsymbol{M}^{\star}}{ds} = \boldsymbol{T}^{\star}. \qquad [1.10]$$

Thus,

$$\delta a = \int_{u_0}^{u_1} \left[\frac{\partial\sqrt{G(\boldsymbol{M}'_u)}}{\partial\boldsymbol{M}'_u}\,\delta\boldsymbol{M}'_u - \lambda(u)\,\frac{\partial F(\boldsymbol{M}(u))}{\partial\boldsymbol{M}}\,\delta\boldsymbol{M} - F(\boldsymbol{M}(u))\,\delta\lambda \right] du,$$

hence, through integration by parts

$$\delta a = \left[\frac{\partial\sqrt{G(\boldsymbol{M}'_u)}}{\partial\boldsymbol{M}'_u}\,\delta\boldsymbol{M} \right]_{u_0}^{u_1} - \int_{u_0}^{u_1} \left\{ \left[\frac{d}{du}\left(\frac{\partial\sqrt{G(\boldsymbol{M}'_u)}}{\partial\boldsymbol{M}'_u} \right) + \lambda(u)\,\frac{\partial F(\boldsymbol{M}(u))}{\partial\boldsymbol{M}} \right] \delta\boldsymbol{M} + F(\boldsymbol{M}(u))\,\delta\lambda \right\} du.$$

Furthermore, $\delta\boldsymbol{M}(u_0) = \delta\boldsymbol{M}(u_1) = 0$ and the first term is zero. We can deduce two possibilities:

– Let us choose $\delta\boldsymbol{M} = \boldsymbol{0}$. For any $\delta\lambda$, $\delta a = 0$. The fundamental lemma of the calculus of variations implies $F(\boldsymbol{M}(u)) = 0$, and we get again the fact that the curve (C) belongs to the surface (S).

– Let us choose $\delta\lambda = \boldsymbol{0}$. For any $\delta\boldsymbol{M}$, $\delta a = 0$. The fundamental lemma of the calculus of variations implies

$$\frac{d}{du}\left(\frac{\partial\sqrt{G(\boldsymbol{M}'_u)}}{\partial\boldsymbol{M}'_u} \right) + \lambda(u)\,\frac{\partial F(\boldsymbol{M}(u))}{\partial\boldsymbol{M}} = 0.$$

Taking into account equation [1.10], we obtain

$$\frac{d\boldsymbol{T}}{du} + \lambda(u)\,\left(\frac{\partial F(\boldsymbol{M}(u))}{\partial\boldsymbol{M}} \right)^{\star} = 0.$$

Noting that $\left(\dfrac{\partial F(\boldsymbol{M}(u))}{\partial \boldsymbol{M}}\right)^{\star} = \mathrm{gradF}$ and that Frenet's first formula is written as $\dfrac{d\boldsymbol{T}}{ds} = \dfrac{\boldsymbol{N}}{\rho}$, where $\boldsymbol{N}$ is the principal normal and ρ is the radius of the algebraic curvature of (C) at $\boldsymbol{M}$, we obtain

$$\frac{\boldsymbol{N}}{\rho} + \lambda(u)\frac{du}{ds}\mathrm{gradF} = 0.$$

Let us write $\mu(u) = -\lambda(u)\dfrac{du}{ds}$gradF, we obtain

$$\boldsymbol{N} = \mu(u)\mathrm{gradF}. \qquad [1.11]$$

As the osculating plane of (C) at $\boldsymbol{M}$ is the plane $\boldsymbol{M}\,\boldsymbol{T}\,\boldsymbol{N}$, we obtain the following theorem.

THEOREM 1.5.– The curves of a surface connecting two points A and B, and whose length is an extremum, are those for which the osculating plane is normal to the surface.

Such curves are called *geodesics of the surface* (see Figure 1.2).

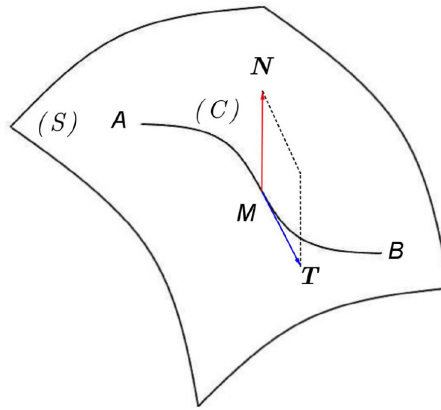


Figure 1.2. *Geodesic curve of the surface F connecting the points $\boldsymbol{A}$ with $\boldsymbol{B}$. The plane $(\boldsymbol{M}, \boldsymbol{T}, \boldsymbol{N})$ represents the osculating plane of the curve*

EXAMPLE 1.3.– Geodesics of a sphere.

Let (S) be a sphere with the center O and radius R_o. At a point M on the sphere, $\boldsymbol{OM}$ is normal to (S) and parallel to $\boldsymbol{N}$. The condition [1.11] is satisfied. Furthermore,

$$\boldsymbol{T} = \frac{d\boldsymbol{OM}}{ds} \quad \text{and} \quad \frac{d\boldsymbol{T}}{ds} = \frac{\boldsymbol{N}}{\rho};$$

thus, $\dfrac{d^2\boldsymbol{OM}}{ds^2}$ is parallel to $\boldsymbol{OM}$. The condition [1.11] is equivalent to

$$\frac{d}{ds}\left(\frac{d\boldsymbol{OM}}{ds} \times \boldsymbol{OM}\right) = \boldsymbol{0} \quad \Longleftrightarrow \quad \frac{d\boldsymbol{OM}}{ds} \times \boldsymbol{OM} = \boldsymbol{K},$$

where $\boldsymbol{K}$ is a constant vector and $\times$ denotes the vector product. $\boldsymbol{OM}$ is orthogonal to $\boldsymbol{K}$, and $\boldsymbol{M}$ belongs to the diametral plane of the sphere (S) orthogonal to $\boldsymbol{K}$; the geodesics of the sphere are portions of its great circles.

2

Variation of Curvilinear Integral

2.1. Geometrization of variational problems

As we have seen in Chapter 1, the mapping from an interval $[t_0, t_1]$ included in $\mathbb{R}$ to $\mathbb{R}^n$, i.e. made up of n real functions,

$$t \in I \equiv [t_0, t_1] \longrightarrow \boldsymbol{M} = \varphi(t) \in \mathbb{R}^n,$$

represents a curve (C) of $\mathbb{R}^n$ associated with parameter t. The mapping φ is differentiable over $[t_0, t_1]$ as many times as necessary.

The search for n real functions leads to the search for a curve in an n-dimensional space, and consequently it is possible to transform a variational problem into a geometric problem. Indeed, it is when the problem statement brings in only quantities attached to the curve – i.e. independent of the parameterization – that the problem is a geometric problem and the calculus of variations' problems are stated in the form of *finding a curve of* $\mathbb{R}^n$ *that realizes an extremum condition.* Consider the differentiable mapping

$$(t, u) \in I \times J \longrightarrow \psi(t, u) \in \mathbb{R}^n,$$

where $J \equiv [u_0, u_1]$ is an interval of $\mathbb{R}$ containing 0 such that $\psi(t, 0) = \varphi(t)$. For each value of u belonging to J, the mapping ψ defines curves $\{(\mathcal{C}_u), u \in J\}$ such that when u has the value 0, $(\mathcal{C}_0) = (\mathcal{C})$. We have defined *two main families of curves*:

$$(\mathcal{C}_u),\ u \in J: \quad t \in I \longrightarrow \psi(t, u) \in \mathbb{R}^n,$$
$$(\Gamma_t),\ t \in I: \quad u \in J \longrightarrow \psi(t, u) \in \mathbb{R}^n.$$

In the particular case where $n = 3$, the mapping ψ defines a surface (S) of $\mathbb{R}^3$. The curves $(\mathcal{C}_u)$ and (Γ_t) are coordinate lines of the surface (S). Each point $M(t,u)$ on the surface (S) is generally determined by the intersection of the two coordinate lines. The curves $(\mathcal{C}_u)$ resulting from the first family are the *deformed shapes* of $(\mathcal{C}_0) = (\mathcal{C})$ through the deformations associated with the second family. The mapping ψ is differentiable, and we note

$$d\boldsymbol{M} = \frac{\partial \boldsymbol{M}(t,u)}{\partial t}\, dt + \frac{\partial \boldsymbol{M}(t,u)}{\partial u}\, du \quad \text{representing}$$
$$d\psi = \frac{\partial \psi(t,u)}{\partial t}\, dt + \frac{\partial \psi(t,u)}{\partial u}\, du.$$

The vector $d\boldsymbol{M}$ belongs to the plane tangent to the surface (S) at $\boldsymbol{M}(t,u)$, for which surface $\dfrac{\partial \psi(t,u)}{\partial t}$ and $\dfrac{\partial \psi(t,u)}{\partial u}$ constitute a tangent-plane basis. Given the requirements of the problem, we simply change the notations such that (see Figure 2.1):

$$d\boldsymbol{M} = \frac{\partial \psi(t,0)}{\partial t}\, dt \text{ and } \delta\boldsymbol{M} = \frac{\partial \psi(t,0)}{\partial u}\, du.$$

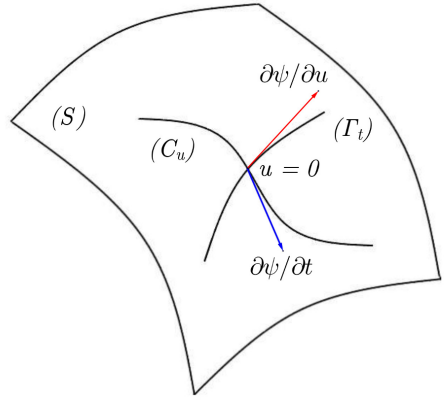


Figure 2.1. *The coordinate curves of the surface (S) associated with two position parameters, u and t. For a color version of the figures in this chapter, see www.iste.co.uk/gouin/mechanics.zip*

The vector $d\boldsymbol{M}$ is the value of the differential of ψ for $du = 0$ taken at a point on $(\mathcal{C})$, i.e. for $u = 0$. The vector $\delta\boldsymbol{M}$ is the value of the differential of ψ for $dt = 0$ taken at a point on $(\mathcal{C})$, i.e. for $u = 0$. We have the important relation

$$d\delta\boldsymbol{M} = \delta d\boldsymbol{M}$$

corresponding to the interchange of the two partial derivatives of the C^2-mapping ψ. Thus, $\delta \boldsymbol{M}$ will be called the *deformation vector field* of the curve $(\mathcal{C})$ associated with the family $\{(\mathcal{C}_u), u \in J\}$.

2.2. First form of curvilinear integral

The vector space $\mathbb{R}^n$ is assumed to be Euclidean. Given a differentiable vector field of $\mathbb{R}^n$,

$$\boldsymbol{M} \in \mathbb{R}^n \longrightarrow \boldsymbol{V} = \Phi(\boldsymbol{M}) \in \mathbb{R}^n,$$

we consider a curve $(\mathcal{C})$ connecting two points A and B. The curve $(\mathcal{C})$ is assumed to be differentiable.

DEFINITION 2.1.– We call circulation of the vector field $\boldsymbol{V}$ the curvilinear integral

$$a = \int_{(\mathcal{C})} \boldsymbol{V}^\star \, d\boldsymbol{M},$$

where $d\boldsymbol{M} = \varphi'(t)\, dt$ with $t \in I \equiv [t_0, t_1] \longrightarrow \boldsymbol{M} = \varphi(t) \in \mathbb{R}^n$ being the representation of the curve $(\mathcal{C})$.

Indeed, we only need a form-field as

$$\boldsymbol{M} \in \mathbb{R}^n \longrightarrow \boldsymbol{V}^\star = \Phi^\star(\boldsymbol{M}) \in \mathbb{R}^{n\star}.$$

This is the case for an Euclidean space; the metric induces a transposition denoted by *, which associates a linear form with each vector. Any form field $\boldsymbol{V}^\star$ is associated with an integral a, which can be written as

$$a = \mathcal{G}(\mathcal{C}).$$

We propose to study the concept of variation of a for the deformation field ψ. To each curve $(\mathcal{C}_u)$, with $u \in J$, represented by

$$t \in I \longrightarrow \boldsymbol{M}(t,u) \equiv \psi(t,u) \in \mathbb{R}^n \text{ with } \psi(t,0) = \varphi(t),$$

is associated the scalar $\mathcal{G}(\mathcal{C}_u)$, which is called the circulation of the form field $\Phi^\star(\boldsymbol{M}) \equiv \boldsymbol{V}^\star(\boldsymbol{M}(t,u))$ along the curve $(\mathcal{C}_u)$ (Figure 2.2):

$$a = \mathcal{G}(\mathcal{C}_u) = \int_{(\mathcal{C}_u)} \boldsymbol{V}^\star(\boldsymbol{M}(t,u) \, \frac{\partial \boldsymbol{M}(t,u)}{\partial t} \, dt.$$

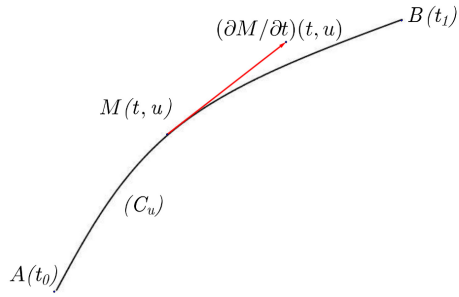


Figure 2.2. *Diagram of the calculus of the curvilinear integral a along the curve $\mathcal{C}_u$*

Thus, for every curve $(\mathcal{C}_u)$, we define a scalar $a = f(u)$. We write $\delta a = f'(u)_{|u=0}\, du$. This expression is analogous to the calculus carried out in Chapter 1. To calculate da, the mapping $\psi(t,u)$ must be twice continuously differentiable,

$$da = \int_{t_0}^{t_1} \left(\frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,u))}{\partial u}\, du\, \frac{\partial \boldsymbol{M}(t,u)}{\partial t} + \boldsymbol{V}^\star(\boldsymbol{M}(t,u))\, \frac{\partial^2 \boldsymbol{M}(t,u)}{\partial t\, \partial u}\, du \right) dt.$$

Through integration by parts, we get

$$da = \left[\boldsymbol{V}^\star(\boldsymbol{M}(t,u))\, \frac{\partial \boldsymbol{M}(t,u)}{\partial u}\, du \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,u))}{\partial u}\, du \right) \frac{\partial \boldsymbol{M}(t,u)}{\partial t}\, dt$$
$$- \int_{t_0}^{t_1} \frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,u))}{\partial t} \left(\frac{\partial \boldsymbol{M}(t,u)}{\partial u}\, du \right) dt.$$

For $u = 0,\ da = \delta a = f'(0)\, du$ and we obtain

$$\delta a = \left[\boldsymbol{V}^\star(\boldsymbol{M}(t,0))\, \frac{\partial \boldsymbol{M}(t,0)}{\partial u}\, du \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,0))}{\partial u}\, du \right) \frac{\partial \boldsymbol{M}(t,0)}{\partial t}\, dt$$
$$- \int_{t_0}^{t_1} \frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,0))}{\partial t} \left(\frac{\partial \boldsymbol{M}(t,0)}{\partial u}\, du \right) dt.$$

But

$$\frac{\partial \boldsymbol{M}(t,0)}{\partial t}\,dt = d\boldsymbol{M} \qquad \text{and} \qquad \frac{\partial \boldsymbol{M}(t,0)}{\partial u}du \;=\; \delta\boldsymbol{M}, \quad \text{so}$$

$$\frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,0))}{\partial t}\,dt = \left(\frac{\partial \boldsymbol{V}(\boldsymbol{M}(t,0))}{\partial \boldsymbol{M}}\frac{\partial \boldsymbol{M}(t,0)}{\partial t}\right)^\star dt = \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\,d\boldsymbol{M}\right)^\star,$$

denoted by $d\boldsymbol{V}^\star$, and

$$\frac{\partial \boldsymbol{V}^\star(\boldsymbol{M}(t,0))}{\partial u}\,du = \left(\frac{\partial \boldsymbol{V}(\boldsymbol{M}(t,0))}{\partial \boldsymbol{M}}\frac{\partial \boldsymbol{M}(t,0)}{\partial u}\right)^\star du = \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\,\delta\boldsymbol{M}\right)^\star,$$

denoted by $\delta\boldsymbol{V}^\star$.

From the diagram of the deformation of $(\mathcal{C}_u)$, shown in Figure 2.3, we deduce the relation that gives the variation of the curvilinear integral in the form

$$\delta a = \Big[\,\boldsymbol{V}^\star\delta\boldsymbol{M}\,\Big]_{\boldsymbol{A}}^{\boldsymbol{B}} + \int_{\widetilde{AB}} \delta\boldsymbol{V}^\star d\,\boldsymbol{M} - \,d\boldsymbol{V}^\star\delta\boldsymbol{M}, \text{ or}$$

$$\delta a = \boldsymbol{V}(\boldsymbol{B})^\star\delta\boldsymbol{B} - \boldsymbol{V}(\boldsymbol{A})^\star\delta\,\boldsymbol{A} + \int_{\widetilde{AB}} \delta\boldsymbol{V}^\star d\,\boldsymbol{M} - \,d\boldsymbol{V}^\star\delta\boldsymbol{M}. \qquad [2.1]$$

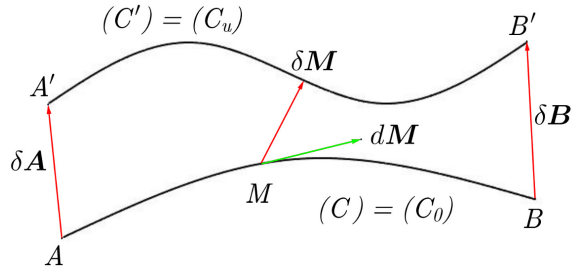


Figure 2.3. *Variation of the curve $(\mathcal{C}_0)$ to $(\mathcal{C}_u)$ by a vector field δM*

Let us explain the calculations. If

$$\boldsymbol{V} \equiv \boldsymbol{V}(\boldsymbol{M}) \text{ with } \quad \boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then

$$\begin{bmatrix} dV_1 \\ dV_2 \\ dV_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \boldsymbol{V}_1}{\partial x} & \dfrac{\partial \boldsymbol{V}_1}{\partial y} & \dfrac{\partial \boldsymbol{V}_1}{\partial z} \\ \dfrac{\partial \boldsymbol{V}_2}{\partial x} & \dfrac{\partial \boldsymbol{V}_2}{\partial y} & \dfrac{\partial \boldsymbol{V}_2}{\partial z} \\ \dfrac{\partial \boldsymbol{V}_3}{\partial x} & \dfrac{\partial \boldsymbol{V}_3}{\partial y} & \dfrac{\partial \boldsymbol{V}_3}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

We write

$$d\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\, d\boldsymbol{M} \quad \text{and} \quad \delta \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\, \delta \boldsymbol{M}.$$

Hence,

$$\delta a = \boldsymbol{V}(\boldsymbol{B})^\star \delta \boldsymbol{B} - \boldsymbol{V}(\boldsymbol{A})^\star \delta \boldsymbol{A} + \int_{\widehat{AB}} \delta \boldsymbol{M}^\star \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^\star d\,\boldsymbol{M} - d\boldsymbol{M}^\star \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^\star \delta \boldsymbol{M},$$

or

$$\delta a = \boldsymbol{V}(\boldsymbol{B})^\star \delta \boldsymbol{B} - \boldsymbol{V}(\boldsymbol{A})^\star \delta \boldsymbol{A} + \int_{\widehat{AB}} \boldsymbol{W}^\star d\boldsymbol{M},$$

$$\text{with } \boldsymbol{W} = \left[\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^\star\right] \delta \boldsymbol{M}.$$

In the case of space $\mathbb{R}^3$,

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^\star = \begin{bmatrix} 0 & -\xi & \eta \\ \xi & 0 & -\zeta \\ -\eta & \zeta & 0 \end{bmatrix} \quad \text{denoted by } i\,(\operatorname{rot} \boldsymbol{V}),$$

where (ζ, η, ξ) are the components of $\operatorname{rot} \boldsymbol{V}$. Then,

$$i(\operatorname{rot} \boldsymbol{V})\, \delta \boldsymbol{M} = \operatorname{rot} \boldsymbol{V} \times \delta \boldsymbol{M}$$

and

$$\delta a = \boldsymbol{V}(\boldsymbol{B})^\star \delta \boldsymbol{B} - \boldsymbol{V}(\boldsymbol{A})^\star \delta\, \boldsymbol{A} + \int_{\widetilde{AB}} (\operatorname{rot} \boldsymbol{V}, \delta \boldsymbol{M}, \mathrm{d}\boldsymbol{M}).$$

This form of writing, restricted to a three-dimensional Euclidean space, cannot be generalized to other dimensions.

2.3. Second form of curvilinear integrals

A curvilinear integral often takes the form

$$a = \int_{\widetilde{AB}} n \, ds \qquad [2.2]$$

where n is a differentiable mapping from $\mathbb{R}^p$ to $\mathbb{R}$

$$M \in \mathbb{R}^p \longrightarrow n(M) \in \mathbb{R},$$

and s is the abscissa of the curve $\widetilde{AB}$. We have $d\boldsymbol{M} = \boldsymbol{T} ds$ with $\boldsymbol{T}^\star \boldsymbol{T} = 1$, where $\boldsymbol{T}$ is the unit vector tangent to the arc $\widetilde{AB}$ oriented in the direction of increasing s. The relations demonstrated in the previous section make it possible to calculate the variation of a. Indeed, let us write $\boldsymbol{V} = n(M)\, \boldsymbol{T}$, which follows

$$a = \int_{\widetilde{AB}} n(M)\, \boldsymbol{T}^\star d\boldsymbol{M} \equiv \int_{\widetilde{AB}} \boldsymbol{V}^\star d\boldsymbol{M}.$$

Hence,

$$\delta a = n(B)\, \boldsymbol{T}^\star_B\, \delta \boldsymbol{B} - n(A)\, \boldsymbol{T}^\star_A\, \delta \boldsymbol{A} + \int_{\widetilde{AB}} \delta \left(n(M)\, \boldsymbol{T}^\star\right) d\boldsymbol{M} \; - d\left(n(M)\, \boldsymbol{T}^\star\right) \delta \boldsymbol{M}.$$

This relation is always satisfied because if $\boldsymbol{M}(t,u)$ represents a deformation of $(\mathcal{C}) = (\mathcal{C}_0)$, then

$$a = \int_{\widetilde{A'B'}} n(\boldsymbol{M}(t,u))\, \boldsymbol{T}^\star(t,u)\, \frac{\partial \boldsymbol{M}(t,u)}{\partial t}\, dt$$

and for $u = 0$, we have

$$\delta \boldsymbol{V}^\star = \frac{\partial \boldsymbol{V}^\star(t,0)}{\partial u}\, du, \quad d\boldsymbol{V}^\star = \frac{\partial \boldsymbol{V}^\star(t,0)}{\partial t}\, dt,$$

and

$$\delta \left(n\, \boldsymbol{T}^\star\right) d\boldsymbol{M} \; = n\, \delta \boldsymbol{T}^\star d\, \boldsymbol{M} + \delta n\, \boldsymbol{T}^\star d\boldsymbol{M}.$$

But $\boldsymbol{T}^\star \boldsymbol{T} = 1$ implies $\delta \boldsymbol{T}^\star \boldsymbol{T} = 0$, and thus $\delta \boldsymbol{T}^\star d\boldsymbol{M} = 0$ and $\delta \left(n\, \boldsymbol{T}^\star\right) d\boldsymbol{M} \; = \delta n\, ds$.

Furthermore, $d\left(n\, \boldsymbol{T}\right) = \left(\dfrac{d\left(n\, \boldsymbol{T}\right)}{ds}\right) ds$. Finally,

$$\delta a = n(B)\, \boldsymbol{T}^\star_B\, \delta \boldsymbol{B} - n(A)\, \boldsymbol{T}^\star_A\, \delta \boldsymbol{A} + \int_{\widetilde{AB}} \left[\delta n - \delta \boldsymbol{M}^\star \frac{dn\boldsymbol{T}}{ds}\right] ds.$$

Since

$$\delta n = \frac{\partial n}{\partial \boldsymbol{M}}\,\delta \boldsymbol{M} = \delta \boldsymbol{M}^{\star}\mathrm{gradn},$$

we can write[1]

$$\delta a = [n\,\boldsymbol{T}^{\star}\delta \boldsymbol{M}]_A^B + \int_{\widetilde{AB}} \left[(\mathrm{gradn}) - \left(\frac{\mathrm{dn}\boldsymbol{T}}{\mathrm{ds}}\right)\right]^{\star} \delta \boldsymbol{M} ds,$$

which gives along the extremals

$$\mathrm{gradn} - \frac{\mathrm{dn}\boldsymbol{T}}{\mathrm{ds}} = 0.$$

This is the most commonly used relation; however, we can also make it explicit in another form

$$\frac{dn\boldsymbol{T}}{ds} = n\frac{d\boldsymbol{T}}{ds} + \boldsymbol{T}\boldsymbol{T}^{\star}\mathrm{gradn} \quad = \mathrm{n}\frac{\boldsymbol{N}}{\mathrm{R}} + \boldsymbol{T}\boldsymbol{T}^{\star}\mathrm{gradn}$$

because

$$\frac{dn}{ds} = \frac{\partial n}{\partial \boldsymbol{M}}\frac{d\boldsymbol{M}}{ds} = (\mathrm{gradn})^{\star}\,\boldsymbol{T} = \boldsymbol{T}^{\star}\mathrm{gradn},$$

where $\boldsymbol{N}$ is the principal normal vector to the curve $\widetilde{AB}$ at M and R is the radius of curvature. So, we obtain

$$\delta a = [n\,\boldsymbol{T}^{\star}\delta \boldsymbol{M}]_A^B + \int_{\widetilde{AB}} \delta \boldsymbol{M}^{\star}\left[(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star})\,\mathrm{gradn} - \mathrm{n}\frac{\boldsymbol{N}}{\mathrm{R}}\right] ds, \qquad [2.3]$$

where $\mathbf{1}$ denotes the identity.

REMARK 2.1.– The term $\boldsymbol{T}\boldsymbol{T}^{\star}$ is represented in orthonormal axes by the $p \times p$ matrix

$$\begin{bmatrix} T_1 \\ \vdots \\ T_p \end{bmatrix} \begin{bmatrix} T_1, \cdots, T_p \end{bmatrix},$$

1 In orthonormal axes,

$$\frac{\partial n}{\partial \boldsymbol{M}} = \left[\frac{\partial n}{\partial x_1}, \ldots, \frac{\partial n}{\partial x_p}\right] = (\mathrm{gradn})^{\star}.$$

which stands for the projection on the tangent to the curve $\widetilde{AB}$ at M.

Indeed, if $\boldsymbol{W}$ is a vector of $\mathbb{R}^p$, $\boldsymbol{T}\boldsymbol{T}^\star W = \boldsymbol{T}\left(\boldsymbol{T}^\star W\right)$ and $\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star$ represents the projection on the plane normal to the curve $\widetilde{AB}$ at M, i.e. normal to the vector $\boldsymbol{T}$ at M

REMARK 2.2.– Instead of using the fundamental formula [2.1] demonstrated in section 2.2, we could carry out a more direct calculation of the variation of the curvilinear integral [2.2]:

Let $\boldsymbol{M}(t,u)$ be the deformation defined earlier. Then,

$$\delta a = \int_{\widetilde{AB}} \delta n\left(M\right)\ ds + n\left(M\right)\ \delta ds.$$

Since $d\boldsymbol{M} = \boldsymbol{T}ds$, we have $ds = \boldsymbol{T}^\star d\boldsymbol{M}$. According to the Schwarz theorem, $\delta d\boldsymbol{M} = d\delta\boldsymbol{M}$ and consequently,

$$\delta ds = \delta\left(\boldsymbol{T}^\star d\boldsymbol{M}\right) = \delta\boldsymbol{T}^\star d\boldsymbol{M} + \boldsymbol{T}^\star \delta d\boldsymbol{M} = \delta\boldsymbol{T}^\star\boldsymbol{T}ds + \boldsymbol{T}^\star d\delta\boldsymbol{M}.$$

But $\boldsymbol{T}^\star\boldsymbol{T} = 1$ implies $\delta\boldsymbol{T}^\star\boldsymbol{T} = \mathbf{0}$ and $\delta ds = \boldsymbol{T}^\star d\delta\boldsymbol{M} = \boldsymbol{T}^\star \dfrac{d\delta\boldsymbol{M}}{ds}\, ds$. It follows

$$\begin{aligned}
\delta a &= \int_{\widetilde{AB}} \left(\frac{\partial n}{\partial \boldsymbol{M}}\ \delta\boldsymbol{M}\ + n\left(M\right)\ \boldsymbol{T}^\star \frac{d\delta\boldsymbol{M}}{ds} \right) ds \\
&= \int_{\widetilde{AB}} \left((\mathrm{gradn})^\star \delta\boldsymbol{M} + \left[\frac{\mathrm{d}\left(\mathrm{n}\boldsymbol{T}^\star\delta\boldsymbol{M}\right)}{\mathrm{ds}} - \left(\frac{\mathrm{dn}\boldsymbol{T}}{\mathrm{ds}} \right)^\star \delta\boldsymbol{M} \right] \right) ds \\
&= \left[n\,\boldsymbol{T}^\star\delta\boldsymbol{M} \right]_A^B + \int_{\widetilde{AB}} \left[\mathrm{gradn} - \left(\frac{\mathrm{dn}\boldsymbol{T}}{\mathrm{ds}} \right) \right]^\star \delta\boldsymbol{M}\, ds
\end{aligned}$$

which corresponds to result [2.3].

2.4. Generalization and variation of derivative

It may happen that the curvilinear integral corresponds not only to the circulation of a vector field defined at each point of the space but also to the circulation of intrinsic quantities defined along the curve such as $\boldsymbol{N}, \boldsymbol{T}, R$. Relation [2.1] in section 2.2 can be applied without any change. However, it is useful to apply a preliminary lemma concerning the variation of derivative.

LEMMA 2.1.– ***Variation of derivative***.

Let the twice continuously differentiable mapping be

$$\boldsymbol{x} \in \mathbb{R}^n \longrightarrow \boldsymbol{A} = F(\boldsymbol{x}) \in \mathbb{R}^p.$$

Then,

$$\delta\left(\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}}\right) = \frac{\partial\, \delta \boldsymbol{A}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}} \frac{\partial\, \delta \boldsymbol{x}}{\partial \boldsymbol{x}}, \qquad [2.4]$$

where $\dfrac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}}$ denotes the tangent linear mapping of $F(\boldsymbol{x})$.

PROOF.– The expression $d\boldsymbol{A} = F'(\boldsymbol{x})\, d\boldsymbol{x}$ is written in matrix formulation

$$d\boldsymbol{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} \text{ vector of } \mathbb{R}^n,$$

$$\text{and } F'(\boldsymbol{x}) = \left[\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_n}\right], \text{ a matrix with } p \text{ lines and } n \text{ columns.}$$

We can choose for $d\boldsymbol{x}$ a mapping $H(\boldsymbol{x})$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. $d\boldsymbol{A} = F'(\boldsymbol{x})\, H(\boldsymbol{x})$; for the sake of convenience, we call $H(\boldsymbol{x})$ the variation of $\boldsymbol{x}$ and we denote it by $\delta\boldsymbol{x}$. Thus, $\boldsymbol{x} \in \mathbb{R}^n \longrightarrow \delta\boldsymbol{x} \in \mathbb{R}^n$ is a vector field of $\mathbb{R}^n$ and similarly, we write $\delta\boldsymbol{A} = F'(\boldsymbol{x})\, \delta\boldsymbol{x} = G(\boldsymbol{x})$. Let us calculate $dG(\boldsymbol{x})$; we have

$$dG(\boldsymbol{x}) = d\, \delta \boldsymbol{A} = dF'(\boldsymbol{x})\, \delta\boldsymbol{x} + F'(\boldsymbol{x})\, d\, \delta\boldsymbol{x}. \qquad [2.5]$$

Furthermore, $dF'(\boldsymbol{x}) = F''(x)\, d\boldsymbol{x}$. The term $F''(x)$ is a convenient operator field. Thus,

$$(F''(x)\, d\boldsymbol{x})\, \delta\boldsymbol{x} = \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i \delta x_j.$$

As the mapping F is twice continuously derivable, $\dfrac{\partial^2 F}{\partial x_i \partial x_j} = \dfrac{\partial^2 F}{\partial x_j \partial x_i}$ and $F''(x)$ is a symmetric bilinear operator,

$$F''(x)\, d\boldsymbol{x}\, \delta\boldsymbol{x} = F''(x)\, \delta\boldsymbol{x}\, d\boldsymbol{x}.$$

Relation [2.5] becomes

$$d\,\delta\boldsymbol{A} = (F''(x)\;\delta\boldsymbol{x})\,d\boldsymbol{x} + F'(\boldsymbol{x})\frac{\partial\delta\boldsymbol{x}}{\partial\boldsymbol{x}}\,d\boldsymbol{x}.$$

Furthermore, $d\,\delta\boldsymbol{A} = \dfrac{\partial\delta\boldsymbol{A}}{\partial\boldsymbol{x}}\,d\boldsymbol{x}$, where $\dfrac{\partial\delta\boldsymbol{A}}{\partial\boldsymbol{x}}$ is the tangent linear operator of the field $G(\boldsymbol{x}) = \delta\boldsymbol{A}$ assumed to be differentiable if $H(\boldsymbol{x}) = \delta\boldsymbol{x}$ is also differentiable. Hence,

$$F''(x)\;\delta\boldsymbol{x} + F'(x)\frac{\partial\delta\boldsymbol{x}}{\partial\boldsymbol{x}} = \frac{\partial\delta\boldsymbol{A}}{\partial\boldsymbol{x}},$$

or

$$\delta F'(x) + F'(x)\frac{\partial\delta\boldsymbol{x}}{\partial\boldsymbol{x}} = \frac{\partial\delta\boldsymbol{A}}{\partial\boldsymbol{x}}.$$

Furthermore, $F'(\boldsymbol{x}) = \dfrac{\partial\boldsymbol{A}}{\partial\boldsymbol{x}}$ and we obtain

$$\frac{\partial\,\delta\boldsymbol{A}}{\partial\boldsymbol{x}} = \delta\left(\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{x}}\right) + \frac{\partial\boldsymbol{A}}{\partial\boldsymbol{x}}\frac{\partial\,\delta\boldsymbol{x}}{\partial\boldsymbol{x}},$$

and we get relation [2.4]. ■

EXAMPLE 2.1.– Determine the variations of

$$a = \int_{(\mathcal{C})} \boldsymbol{V}^\star(\boldsymbol{T})\,d\boldsymbol{M},$$

where $\boldsymbol{V}^\star$ is a function of the unit tangent vector along the curve $(\mathcal{C})$.

The calculations are as follows:

$$\delta a = \Big[\,\boldsymbol{V}^\star\delta\boldsymbol{M}\,\Big]_A^B + \int_{\widetilde{AB}} \delta\boldsymbol{V}^\star d\boldsymbol{M} - d\boldsymbol{V}^\star\delta\boldsymbol{M} \quad \text{with} \quad \delta\boldsymbol{V} = \frac{\partial\boldsymbol{V}}{\partial\boldsymbol{T}}\,\delta\boldsymbol{T}.$$

We must calculate $\delta\boldsymbol{T}$ as a function of $\delta\boldsymbol{M}$.

According to Lemma 2.1,

$$\boldsymbol{T} = \frac{d\boldsymbol{M}}{ds} \quad\Longrightarrow\quad \delta\boldsymbol{T} = \boldsymbol{\delta}\left(\frac{d\boldsymbol{M}}{ds}\right) = \frac{d\delta\boldsymbol{M}}{ds} - \frac{d\boldsymbol{M}}{ds}\frac{d\delta s}{ds}.$$

According to the Schwarz theorem,

$$d\delta s = \delta ds \quad \text{and} \quad \delta d\boldsymbol{M} = d\delta \boldsymbol{M};$$

hence,

$$d\delta s = \delta\left(\boldsymbol{T}^{\star} d\boldsymbol{M}\right) = \delta \boldsymbol{T}^{\star} d\boldsymbol{M} + \boldsymbol{T}^{\star} \delta d\boldsymbol{M} = \boldsymbol{T}^{\star} d\delta \boldsymbol{M},$$

because

$$\delta \boldsymbol{T}^{\star} d\boldsymbol{M} = \delta \boldsymbol{T}^{\star} \boldsymbol{T} ds = 0.$$

Therefore,

$$\delta \boldsymbol{T} = \frac{d\delta \boldsymbol{M}}{ds} - \frac{d\boldsymbol{M}}{ds}\frac{d\delta s}{ds} = \left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\frac{d\delta \boldsymbol{M}}{ds}.$$

Furthermore,

$$\delta \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\,\delta \boldsymbol{T} \quad \Longrightarrow \quad \delta \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\frac{d\delta \boldsymbol{M}}{ds}.$$

Hence,

$$\begin{aligned}\delta \boldsymbol{V}^{\star} d\boldsymbol{M} &= \left(\frac{d\delta \boldsymbol{M}}{ds}\right)^{\star}\left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^{\star} \boldsymbol{T} ds \\ &= \left(d\delta \boldsymbol{M}\right)^{\star}\left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^{\star} \boldsymbol{T}.\end{aligned}$$

We obtain

$$\delta a = \left[\, \boldsymbol{V}^{\star} \delta \boldsymbol{M} \,\right]_A^B + \int_{\widetilde{AB}} \left(\left(d\delta \boldsymbol{M}\right)^{\star}\left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^{\star} \boldsymbol{T} - d\boldsymbol{V}^{\star} \delta \boldsymbol{M} \right)$$

and through integration by parts,

$$\begin{aligned}\delta a = &\left[\, \boldsymbol{V}^{\star} \delta \boldsymbol{M} + \delta \boldsymbol{M}^{\star}\left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^{\star} \boldsymbol{T} \,\right]_A^B \\ &- \int_{\widetilde{AB}} \delta \boldsymbol{M}^{\star} d\left[\, \boldsymbol{V} + \left(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}\right)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^{\star} \boldsymbol{T} \,\right],\end{aligned}$$

or

$$\delta a = \left[\left(\boldsymbol{V}^\star + \boldsymbol{T}^\star\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)(\mathbf{1}-\boldsymbol{T}\boldsymbol{T}^\star)\right)\delta \boldsymbol{M}\right]_A^B - \int_{\widetilde{AB}} \delta \boldsymbol{M}^\star \frac{d}{ds}\left[\boldsymbol{V}+(\mathbf{1}-\boldsymbol{T}\boldsymbol{T}^\star)\left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}}\right)^\star \boldsymbol{T}\right] ds,$$

which fits with the case $\boldsymbol{V}(\boldsymbol{T}) = \boldsymbol{T}$ corresponding to constant scalar n.

2.5. First application: studying the optical path of light

2.5.1. *Fermat's principle*

The notations used are those given in section 1.6.4 in Chapter 1. In the real three-dimensional Euclidean space $\mathcal{E}^3$ corresponding to the usual physical space, and using the direct orthonormal coordinate system $\boldsymbol{O\,ijk}$, we consider a differentiable mapping

$$M \in \mathcal{E}^3 \longrightarrow n(\boldsymbol{M}) \in \mathbb{R},$$

called the refractive index of the *medium*.

PRINCIPLE 2.1.– In a medium whose refractive index is $n(\boldsymbol{M})$, the trajectory of the ray of light going from point A to point B is among the curves connecting A and B, the curve which makes extremal

$$a = \int_{(C)} n(\boldsymbol{M})\, ds \equiv \int_{(C)} n(\boldsymbol{M})\, \boldsymbol{T}^\star d\boldsymbol{M}.$$

The trajectory is called the optical path.

REMARK 2.3.– Given a curve (C) defined by the differentiable mapping

$$t \in I \longrightarrow \varphi(t) \equiv \boldsymbol{M}(t) \in \mathbb{R}^3,$$

and given a vector field defined at any point along the curve (C)

$$M \in (C) \longrightarrow \boldsymbol{H}(\boldsymbol{M}),$$

a ruled surface is defined by the mapping (see Figure 2.4)

$$(t,u) \in I \times \mathbb{R} \longrightarrow \psi(t,u) = \boldsymbol{M}(t) + u\, \boldsymbol{H}(\boldsymbol{M}(t))$$

We note that $\psi(t,0)=\varphi(t)$. So, we have defined a deformation field ψ such that

$$\delta \boldsymbol{M} = \frac{\partial \psi(t,u)}{\partial u}\Big|_{u=0} du = \boldsymbol{H}(\boldsymbol{M}(t))\, du; \text{ we can choose } du = 1,$$
$$d\boldsymbol{M} = \frac{\partial \psi(t,u)}{\partial t}\Big|_{u=0} dt = \varphi'(t)\, dt; \text{ we can choose } dt = 1.$$

Thus, any arbitrary differentiable field $\boldsymbol{H}(\boldsymbol{M})$ defined at any point on (C) can be considered as a field $\delta\boldsymbol{M}$ associated with a deformation ψ of (C).

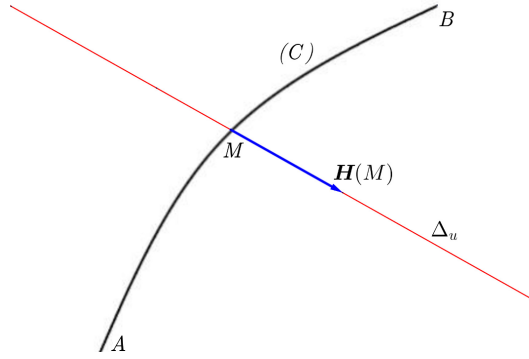


Figure 2.4. *Variation of the curve (C) generating a ruled surface formed by the lines carried by the vector field $\boldsymbol{H}(\boldsymbol{M}(t))$*

Let us return to Fermat's principle. The single-parameter families associated with a variation of (C) are made up of curves connecting A and B. That yields $\delta\boldsymbol{A}=0$ and $\delta\boldsymbol{B}=0$, and for any $\delta\boldsymbol{M}$, we can write

$$\delta a = \int_{(C)} \delta\boldsymbol{M}^{\star}\left[(\mathbf{1}-\boldsymbol{T}\boldsymbol{T}^{\star})\,\mathrm{gradn} - \mathrm{n}\frac{\boldsymbol{N}}{\mathrm{R}}\right] ds \;=0$$

The field $\delta\boldsymbol{M}$ can be chosen as any continuous field of vectors that are null at A and B. According to the lemma in Chapter 1, section 1.1 on the calculus of variations, Fermat's principle implies

$$(\mathbf{1}-\boldsymbol{T}\boldsymbol{T}^{\star})\,\mathrm{gradn} - \mathrm{n}\frac{\boldsymbol{N}}{\mathrm{R}} = \mathbf{0}.$$

Consider the Frenet frame $M\boldsymbol{T}\boldsymbol{N}\boldsymbol{B}$; therefore, $(\mathbf{1}-\boldsymbol{T}\boldsymbol{T}^{\star})$ is the projection on the normal plane of the Frenet frame (see Figure 2.5). So, the projection of gradn is

carried by the principal normal direction to the optical path. Consequently, gradn is in the plane $M\,\boldsymbol{TN}$, i.e. it belongs to the osculation plane to the optical path at M. Multiplying the above relation by $\boldsymbol{N}^\star$, we deduce

$$\mathrm{R}\,\boldsymbol{N}^\star\,\frac{\mathrm{gradn}}{n}=1.$$

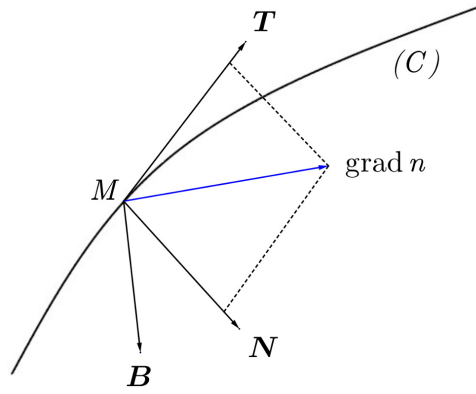


Figure 2.5. *Frenet frame $M\,\boldsymbol{TNB}$ of the curve $(\mathcal{C})$ at M*

Let point D such that $\boldsymbol{MD}$ = $\mathrm{R}\,\boldsymbol{N}$ and I be the orthogonal projection of K on $\boldsymbol{MD}$ such that $\boldsymbol{MK}=\dfrac{\mathrm{gradn}}{n}$, and H be the orthogonal projection of D on $\boldsymbol{MK}$. In the plane $M\,D\,H$, we have

$$\frac{\overline{MI}}{\overline{MH}}=\frac{\overline{MK}}{\overline{MD}}\quad\text{or}\quad\overline{MI}\,\overline{MD}\,=\overline{MK}\,\overline{MH}.$$

Furthermore, $\overline{MD}\,=\mathrm{R}$ and $\overline{MI}\,=\boldsymbol{N}^\star\,\dfrac{\mathrm{gradn}}{n}$, which yields $\overline{MI}\,\overline{MD}\,=1$. Consequently,

$$\overline{MK}=\frac{\|\mathrm{gradn}\|}{n}\implies\overline{MH}=\frac{n}{\|\mathrm{gradn}\|}.$$

The center of curvature of the optical path at M can be found in the plane (Π) perpendicular to the vector gradn and at a distance MH such that

$$MH=\frac{n}{\|\mathrm{gradn}\|}.$$

2.5.2. *Descartes' laws*

These are the laws that govern the optical path of a diopter. Consider a surface (S) in $\mathcal{E}^3$ which separates the physical space between two regions $\mathcal{E}_1$ and $\mathcal{E}_2$ whose refractive indices are n_1 and n_2, respectively. These refractive indices are assumed to be differentiable, and for $M \in (S)$, $n_1(\boldsymbol{M}) \neq n_2(\boldsymbol{M})$. It is assumed that $n_1(\boldsymbol{M}_1)$, $\boldsymbol{M}_1 \in \mathcal{E}_1$ and $n_2(\boldsymbol{M}_2)$, $\boldsymbol{M}_2 \in \mathcal{E}_2$, admit a limit when M_1 and M_2 tend towards any point M of (S). We say that (S) makes up a diopter. We write

$$[n(\boldsymbol{M})] = \lim_{M_1 \to M \in (S)} n_1(\boldsymbol{M}_1) - \lim_{M_2 \to M \in (S)} n_2(\boldsymbol{M}_2).$$

Since A and B are the two ends of the optical path (C), let us write

$$a = \int_{\widetilde{AD}} n_1(\boldsymbol{M})\, ds + \int_{\widetilde{DB}} n_2(\boldsymbol{M})\, ds,$$

where $D = (C) \cap (S)$.

Let us assume that the optical path (C) is differentiable out of the diopter (S); then,

$$\begin{aligned}
\delta a = {} & n_1(D)\, \boldsymbol{T}_1^\star(D)\, \delta \boldsymbol{D} - n_1(A)\, \boldsymbol{T}^\star(A)\, \delta \boldsymbol{A} \\
& + \int_{\widetilde{AD}} \delta \boldsymbol{M}^\star \left[(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)\, \mathrm{gradn}_1 - \mathrm{n}_1 \frac{\boldsymbol{N}}{\mathrm{R}} \right] ds \\
& + n_2(B)\, \boldsymbol{T}^\star(B)\, \delta \boldsymbol{B} - n_2(D)\, \boldsymbol{T}_2^\star(D)\, \delta \boldsymbol{D} \\
& + \int_{\widetilde{DB}} \delta \boldsymbol{M}^\star \left[(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)\, \mathrm{gradn}_2 - \mathrm{n}_2 \frac{\boldsymbol{N}}{\mathrm{R}} \right] ds,
\end{aligned} \qquad [2.6]$$

with $\delta \boldsymbol{A} = \mathbf{0}$ and $\delta \boldsymbol{B} = \mathbf{0}$, where $\boldsymbol{T}_1(D)$ and $\boldsymbol{T}_2(D)$ denote the limits on (S) of the vector $\boldsymbol{T}$ tangent to (C) in $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Let us consider the field $\delta \boldsymbol{M}$ satisfying $\delta \boldsymbol{D} = \mathbf{0}$; then,

$$\begin{aligned}
\delta a = {} & \int_{\widetilde{AD}} \delta \boldsymbol{M}^\star \left[(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)\, \mathrm{gradn}_1 - \mathrm{n}_1 \frac{\boldsymbol{N}}{\mathrm{R}} \right] ds \\
& + \int_{\widetilde{AD}} \delta \boldsymbol{M}^\star \left[(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)\, \mathrm{gradn}_2 - \mathrm{n}_2 \frac{\boldsymbol{N}}{\mathrm{R}} \right] ds.
\end{aligned}$$

Thus, for the two regions $\mathcal{E}_1$ and $\mathcal{E}_2$,

$$(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)\, \mathrm{gradn_i} - \mathrm{n_i} \frac{\boldsymbol{N}}{\mathrm{R}} = \mathbf{0} \quad \text{where } \mathrm{i} = 1, 2. \qquad [2.7]$$

By returning to relation [2.6] and taking into account equation [2.7], for the optical path and for any vector field $\delta \boldsymbol{M}$ such as with $\delta \boldsymbol{A} = \boldsymbol{0}$ and $\delta \boldsymbol{B} = \boldsymbol{0}$, we obtain

$$\delta a = [n_1(D)\, \boldsymbol{T}_1^\star(D) - n_2(D)\, \boldsymbol{T}_2^\star(D)]\ \delta \boldsymbol{D}.$$

Thus, for any vector $\delta \boldsymbol{D}$, Fermat's principle implies

$$\delta a = [n_1(D)\, \boldsymbol{T}_1^\star(D) - n_2(D)\, \boldsymbol{T}_2^\star(D)]\ \delta \boldsymbol{D} = 0.$$

Since D belongs to (S), we deduce that $\boldsymbol{N}_S^\star(D)\ \delta \boldsymbol{D} = 0$, where $\boldsymbol{N}_S$ denotes the normal to the surface (S) at D. Consequently,

$$\boldsymbol{N}_S^\star\ \delta \boldsymbol{D} = 0 \quad \Longrightarrow \quad [n_1(D)\, \boldsymbol{T}_1^\star(D) - n_2(D)\, \boldsymbol{T}_2^\star(D)]\ \delta \boldsymbol{D} = 0.$$

This property corresponds to the existence of a real-scalar Lagrange multiplier λ such that

$$\begin{aligned} &n_1(D)\, \boldsymbol{T}_1^\star(D) - n_2(D)\, \boldsymbol{T}_2^\star(D) = \lambda\, \boldsymbol{N}_S^\star(D) \quad \Longleftrightarrow \quad \text{at } D, \\ &n_1\, \boldsymbol{T}_1 - n_2\, \boldsymbol{T}_2 = \lambda\, \boldsymbol{N}_S. \end{aligned}$$

The vectors $\boldsymbol{T}_1, \boldsymbol{T}_2$ and $\boldsymbol{N}_S$ belong to the same plane (Π), and vector $\boldsymbol{u}$ denotes the axis of the tangent plane to (S), intersecting with (Π) such that $(\boldsymbol{u}, \boldsymbol{N}_S) = \pi/2$. By writing $i_p = (\boldsymbol{N}_S, \boldsymbol{T}_p)$, $p = \{1, 2\}$, we obtain

$$n_1\ \sin\ i_1 = n_2\ \sin\ i_2.$$

We can state Descartes' two laws, depicted in Figure 2.6:

a) the incidence plane of the optical path is the same as the reflection plane;

b) $n_1\ \sin\ i_1 = n_2\ \sin\ i_2$.

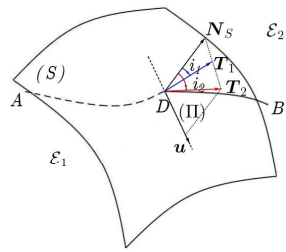


Figure 2.6. *The diopter (S) separates the two media $\mathcal{E}_1$ and $\mathcal{E}_2$. The tangents $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and the normal to the diopter $\boldsymbol{N}_S$ at M belong to the same plane (Π) normal to the surface (S)*

2.6. Second application: the problem of isoperimeters

Consider a differentiable closed plane curve $(\mathcal{C})$, without double point, encircling a quarrable domain $(\mathcal{D})$ (see Figure 2.7). The plane is assumed to be plunged into the oriented Euclidean space $\mathcal{E}^3$, and the length of the curve $(\mathcal{C})$ is $L = \int_{(\mathcal{C})} ds$, where s denotes the positively oriented curvilinear abscissa of $(\mathcal{C})$. The area of the domain $(\mathcal{D})$ is $\mathcal{A} = \iint_{(\mathcal{D})} d\sigma$, where $d\sigma$ denotes the area measurement in the plane.

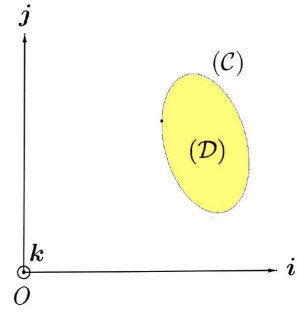


Figure 2.7. *Quarrable domain $(\mathcal{D})$ of the curve $(\mathcal{C})$ without double point and for which the origin and extremity are confounded*

We deduce

$$L = \int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M} \quad \text{and} \quad \mathcal{A} = \iint_{(\mathcal{D})} \boldsymbol{n}^\star \boldsymbol{n}\, d\sigma,$$

where $\boldsymbol{n}$ denotes the unit vector orthogonal to the Euclidean plane containing the curve (C). In orthonormal axes $\boldsymbol{i}, \boldsymbol{j}$ of the plane, the vector $\boldsymbol{n} = \boldsymbol{i} \times \boldsymbol{j} \equiv \boldsymbol{k}$ completes the orthonormal trihedron in the space and forms the right-handed orthonormal coordinate system of $\mathcal{E}^3$. There exists a vector field $\boldsymbol{V} \equiv X(x,y,z)\,\boldsymbol{i} + Y(x,y,z)\,\boldsymbol{j} + Z(x,y,z)\,\boldsymbol{n}$ such that

$$\mathrm{rot}\boldsymbol{V} = \begin{bmatrix} \dfrac{\partial Z}{\partial y} - \dfrac{\partial Y}{\partial z} \\ \dfrac{\partial X}{\partial z} - \dfrac{\partial Z}{\partial x} \\ \dfrac{\partial Y}{\partial x} - \dfrac{\partial X}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad [2.8]$$

For the solution of this system of partial differential equations [2.8], we choose $X = -y/2,\ Y = x/2$ and $Z = 0$, and we can write

$$\mathcal{A} = \iint_{(\mathcal{D})} (\mathrm{rot}\boldsymbol{V})^\star\, \boldsymbol{n}\, \mathrm{d}\sigma.$$

The area $\mathcal{A}$ represents the flow of $\mathrm{rot}\boldsymbol{V}$ through the domain $(\mathcal{D})$ oriented by the vector $\boldsymbol{n}$, and according to Stokes' theorem,

$$\iint_{(\mathcal{D})} (\mathrm{rot}\boldsymbol{V})^\star \boldsymbol{n}\, \mathrm{d}\sigma = \int_{(\mathcal{C})} \boldsymbol{V}^\star \mathrm{d}\boldsymbol{M},$$

with

$$d\boldsymbol{M} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \quad \text{and} \quad \int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M} = \frac{1}{2} \int_{(\mathcal{C})} x\, dy - y\, dx.$$

This is a well-known result giving the oriented area bounded by a closed-plane curve.

The following two problems are equivalent:

(A) determining the curves of extremal length surrounding a domain with a given area;

(B) determining the curves with a given length surrounding a domain of an extremal area.

Problem *(A)* is equivalent to finding the extremals of

$$L - \lambda \left(\int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M} - \mathcal{A} \right),$$

where λ is a given real. These are the extremals of

$$\int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M} - \lambda \int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M}.$$

Problem *(B)* is equivalent to finding the extremals of

$$\mathcal{A} - \mu \left(\int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M} - L \right),$$

where μ is a given real. These are the extremals of

$$\int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M} - \mu \int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M}.$$

Since

$$\int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M} - \lambda \int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M} = -\lambda \left[\int_{(\mathcal{C})} \boldsymbol{V}^\star d\boldsymbol{M} - \mu \int_{(\mathcal{C})} \boldsymbol{T}^\star d\boldsymbol{M} \right]$$

with

$$\mu = \frac{1}{\lambda},$$

that proves the equivalence.

Let us write

$$a = \int_{(\mathcal{C})} \left(\boldsymbol{T}^\star - \lambda \boldsymbol{V}^\star \right) d\boldsymbol{M}.$$

Since the curve is closed, $A \equiv B$ and the terms integrated by parts in equation [2.1] are zero,

$$\delta a = \int_{(\mathcal{C})} \delta \left(\boldsymbol{T}^\star - \lambda \boldsymbol{V}^\star \right) d\boldsymbol{M} - d \left(\boldsymbol{T}^\star - \lambda \boldsymbol{V}^\star \right) \delta \boldsymbol{M}.$$

Let us note that while $\boldsymbol{V}$ is a vector field, this is not the same for $\boldsymbol{T}$ which is only defined along the curve (C). So, we can consider $\dfrac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}$ but not $\dfrac{\partial \boldsymbol{T}}{\partial \boldsymbol{M}}$. Nonetheless, $\delta \boldsymbol{T}^\star d\boldsymbol{M} = \delta \boldsymbol{T}^\star \boldsymbol{T} ds = 0$, and we deduce

$$\delta a = \int_{(\mathcal{C})} \lambda \left(- \delta \boldsymbol{V}^\star d\boldsymbol{M} + d\boldsymbol{V}^\star \delta \boldsymbol{M} \right) - d\boldsymbol{T}^\star \delta \boldsymbol{M}.$$

Let us note that

$$\begin{aligned} - \delta \boldsymbol{V}^\star d\boldsymbol{M} + d\boldsymbol{V}^\star \delta \boldsymbol{M} &= - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} \delta \boldsymbol{M} \right)^\star d\boldsymbol{M} + \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} d\boldsymbol{M} \right)^\star \delta \boldsymbol{M} \\ &= - \delta \boldsymbol{M}^\star \left[\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} \right)^\star \right]^\star d\boldsymbol{M} \end{aligned}$$

because $\left[\dfrac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\dfrac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^{\star}\right]$ is an anti-symmetrical matrix.

Since $d\boldsymbol{M} = \boldsymbol{T}\, ds$ and $d\boldsymbol{T} = \dfrac{\boldsymbol{N}}{\mathrm{R}}\, ds$, we finally obtain

$$\delta a = \int_{(\mathcal{C})} - \delta \boldsymbol{M}^{\star} \left[\lambda \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^{\star}\right) \boldsymbol{T} - \frac{\boldsymbol{N}}{\mathrm{R}}\right] ds.$$

For any vector field $\boldsymbol{M} \in \mathbb{R}^3 \longrightarrow \delta \boldsymbol{M}$, $\delta a = 0$ and Lemma 1.1 in section 1.3 of Chapter 1 (*the fundamental lemma of the calculus of variations*) implies

$$\lambda \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^{\star}\right) \boldsymbol{T} - c\, \boldsymbol{N} = 0, \quad \text{with } c \equiv \frac{1}{\mathrm{R}}. \tag{2.9}$$

Since

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^{\star} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{then} \quad \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}} - \left(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{M}}\right)^{\star}\right) \boldsymbol{T} = \boldsymbol{n} \times \boldsymbol{T} = \boldsymbol{N},$$

equation [2.9] writes

$$\lambda\, \boldsymbol{N} - c\, \boldsymbol{N} = 0 \;\text{ and }\; c = \lambda\,.$$

The curvature c of (C) is constant. The required plane curves are circles.

3

The Noether Theorem

3.1. Additional results on differential equations

DEFINITION 3.1.– Let

$$(t, x_1, \cdots, x_n) \in \mathcal{O} \subset \mathbb{R}^{n+1} \longrightarrow f(t, x_1, \cdots, x_n) \in \mathbb{R}$$

be a mapping defined on an open set $\mathcal{O}$ of $\mathbb{R}^{n+1}$. The differential equation written as

$$x^{(n)} = f(t, x, x', \cdots, x^{(n-1)}) \qquad [3.1]$$

is called the normal form of the n^{th}-order differential equation. The mapping

$$t \in I \subset \mathbb{R} \longrightarrow x = g(t) \in \mathbb{R}$$

where I is an interval in $\mathbb{R}$, assumed to be n-times derivable and also denoted by x, which is said to be a solution of [3.1] if and only if for any t,

$$g^{(n)}(t) = f(t, g(t), g'(t), \cdots, g^{(n-1)}(t)).$$

For the sake of simplicity, we will note $x(t)$ for $g(t)$. Let us note that if we write $x_1 = x$, $x_2 = x', \cdots, x_n = x^{(n-1)}$, then the differential equation [3.1] can be written as

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad \vdots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, \cdots, x_n) \end{cases} \qquad [3.2]$$

In a coordinate system with the origin O, a point with the coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$ is denoted by M; the vector $\boldsymbol{M}$ (or the bipoint OM) is represented by

$$\boldsymbol{M} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We can write

$$\psi(t, \boldsymbol{M}) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f(t, x_1, \cdots, x_n) \end{bmatrix}, \quad \psi \text{ being an application from } \mathbb{R}^{n+1} \text{ to } \mathbb{R}^n.$$

Equation [3.2] can be written as

$$\frac{d\boldsymbol{M}}{dt} = \psi(t, \boldsymbol{M}). \qquad [3.3]$$

Any system of differential equations can be written in the form [3.3]. Let us recall that if $x(t)$ is a solution defined over the interval I of the differential equation [3.3], its restriction in any subinterval of I is also a solution of [3.3]. Conversely, there may exist solutions of [3.3] that extend $x(t)$. If the only solution extending $x(t)$ is itself, we say that $x(t)$ is a *maximal solution* of [3.3]. This concept can be easily extended to a system of differential equations.

DEFINITION 3.2.– Let $(t, \boldsymbol{M}) \longrightarrow \psi(t, \boldsymbol{M})$ denote a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ and $f_1(t, x_1, \cdots, x_n), \cdots, f_n(t, x_1, \cdots, x_n)$ denote the values of the component mappings constituting ψ, i.e.

$$(t, \boldsymbol{M}) \equiv (t, x_1, \cdots, x_n) \longrightarrow \psi(t, \boldsymbol{M}) = \begin{bmatrix} f_1(t, x_1, \cdots, x_n) \\ \vdots \\ f_n(t, x_1, \cdots, x_n) \end{bmatrix}.$$

The vector equation [3.3] is called a system of first-order differential equations.

THEOREM 3.1.– Let U be an open set of $\mathbb{R} \times \mathbb{R}^n$ and $(t, \boldsymbol{M}) \in U \longrightarrow \psi(t, \boldsymbol{M})$ be a continuous mapping made up of elements $f_1, \cdots, f_n$. It is assumed that $\dfrac{\partial f_i}{\partial x_j}$, where $i, j \in \{1. \cdots, n\}$, exist and are continuous in U. For any point $(t_0, x_{10}, \cdots, x_{n0})$ in U, there exists a maximal solution $(x_1(t), \cdots, x_n(t))$ such that $x_1(t_0) = x_{10}, \cdots, x_n(t_0) = x_{n0}$.

REMARK 3.1.– We are back to $u = 0$ by writing $u = t - t_0$, and equation [3.3] is written in the form

$$\frac{d\boldsymbol{M}}{du} = \psi(u, \boldsymbol{M}).$$

In the specific case where ψ is independent of u, the equation is said to be autonomous[1].

Going forward, we will only consider the case where ψ is independent of u and we write

$$\frac{d\boldsymbol{M}}{du} = W(\boldsymbol{M}). \qquad [3.4]$$

3.2. One-parameter groups and Lie groups

Let $\boldsymbol{M}_0 = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix}$ be a point in $\mathbb{R}^n$. The solution of [3.4] that takes the value $\boldsymbol{M}_0$ for $u = 0$ is denoted by

$$\boldsymbol{M} = \mathcal{T}_u(\boldsymbol{M}_0).$$

For $u = 0$, we have $\boldsymbol{M}_0 = \mathcal{T}_0(\boldsymbol{M}_0)$; thus, $\mathcal{T}_0 = \mathbf{1}$, where $\mathbf{1}$ is the identity.

---

1 Any differential equation of form [3.3] can be considered as an autonomous equation. Indeed, consider the system of differential equations

$$\frac{d\boldsymbol{P}}{du} = \phi(u, \boldsymbol{P}), \text{ where } \boldsymbol{P} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

Writing $u = t$, then $\frac{du}{dt} = 1$ and $\frac{d\boldsymbol{P}}{du} = \frac{d\boldsymbol{P}}{dt}$. Let us write

$$\boldsymbol{M} = \begin{bmatrix} u \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} u \\ \boldsymbol{P} \end{bmatrix}, \text{ with } \frac{d\boldsymbol{P}}{dt} = \phi(u, \boldsymbol{P}), \text{ and } \frac{du}{dt} = 1,$$

that is,

$$W(\boldsymbol{M}) = \begin{bmatrix} \phi(u, \boldsymbol{P}) \\ 1 \end{bmatrix}.$$

Thus, W is a vector field of $\mathbb{R}^{n+1}$ and $\frac{d\boldsymbol{M}}{dt} = W(\boldsymbol{M})$.

THEOREM 3.2.– The transformations $\mathcal{T}_u, u \in I \subset \mathbb{R}$ associated with differential equation [3.4] and satisfying the conditions of Theorem 3.1 satisfy $\mathcal{T}_{u_1} \circ \mathcal{T}_{u_2} = \mathcal{T}_{u_1+u_2}$ and $(\mathcal{T}_u)^{-1} = \mathcal{T}_{(-u)}$.

The transformations $\mathcal{T}_u$, $u \in I \subset \mathbb{R}$ form a group that is called a one-parameter group. We say that $\lim_{u\to 0} \mathcal{T}_u = \mathbf{1}$ if and only if for any $\boldsymbol{M} \in \mathbb{R}^n$, $\lim_{u\to 0} \mathcal{T}_u(\boldsymbol{M}) = \boldsymbol{M}$.

DEFINITION 3.3.– A ***Lie group*** or a continuous one-parameter group is a group $\{\mathcal{T}_u,\ u \in I \subset \mathbb{R}\}$ of transformations of $\mathbb{R}^n$ such that

$$\mathcal{T}_{u_1} \circ \mathcal{T}_{u_2} = \mathcal{T}_{u_1+u_2}, \qquad (\mathcal{T}_u)^{-1} = \mathcal{T}_{(-u)} \qquad \text{and} \qquad \lim_{u\to 0} \mathcal{T}_u = \mathbf{1}.$$

Consequently, $\mathcal{T}_0 = \mathbf{1}$.

EXAMPLE 3.1.– Consider the transformation of the plane

$$\begin{cases} x = x_0 \cos u - y_0 \sin u \\ y = x_0 \sin u + y_0 \cos u \end{cases} \quad \text{thus} \quad \begin{cases} \dfrac{dx}{du} = -y, \\ \dfrac{dy}{du} = x\,. \end{cases}$$

The mapping

$$\boldsymbol{M} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \longrightarrow W(x,y) = \begin{bmatrix} -y \\ x \end{bmatrix} \in \mathbb{R}^2$$

is such that $\dfrac{d\boldsymbol{M}}{du} = W(\boldsymbol{M})$. Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{T}_u \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \text{where} \quad \mathcal{T}_u = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}$$

represents the rotation of the angle u. Indeed, we have

$$\mathcal{T}_{u_1} \circ \mathcal{T}_{u_2} = \mathcal{T}_{u_1+u_2}, \qquad [\mathcal{T}_u]^{-1} = \mathcal{T}_{(-u)} \qquad \text{and} \qquad \lim_{u\to 0} \mathcal{T}_u = \mathbf{1}.$$

DEFINITION 3.4.– A Lie group $\{\mathcal{T}_u,\ u \in I \subset \mathbb{R}\}$ is said to be differentiable if and only if for any $\boldsymbol{M}_0$ in $\mathbb{R}^n$,

$$\frac{\mathcal{T}_{u+h}(\boldsymbol{M}_0) - \mathcal{T}_u(\boldsymbol{M}_0)}{h}$$

admits a limit when h tends to 0 ($h \neq 0$). We write

$$\frac{d\mathcal{T}_u(\boldsymbol{M}_0)}{du} \equiv \mathcal{T}_u'(\boldsymbol{M}_0) = \lim_{h\to 0} \frac{\mathcal{T}_{u+h}(\boldsymbol{M}_0) - \mathcal{T}_u(\boldsymbol{M}_0)}{h}.$$

REMARK 3.2.– Write $\boldsymbol{M} = \mathcal{T}_u(\boldsymbol{M}_0)$. Then,

$$\frac{\mathcal{T}_{u+h}(\boldsymbol{M}_0) - \mathcal{T}_u(\boldsymbol{M}_0)}{h} = \frac{(\mathcal{T}_h - \mathbf{1})(\boldsymbol{M})}{h}.$$

The existence of a limit when h tends to zero can be denoted

$$\left(\frac{d\mathcal{T}_u(\boldsymbol{M})}{du}\right)_{|u=0} \equiv \mathcal{T}_0'(\boldsymbol{M}) = \lim_{h\to 0} \frac{(\mathcal{T}_h - \mathbf{1})(\boldsymbol{M})}{h},$$

or

$$\left(\frac{d\mathcal{T}_u}{du}\right)_{|u=0} \equiv \mathcal{T}_0'.$$

Then,

$$\frac{d\mathcal{T}_u(\boldsymbol{M}_0)}{du} = \mathcal{T}_0' \circ \mathcal{T}_u(\boldsymbol{M}_0).$$

and consequently,

$$\frac{d\boldsymbol{M}}{du} = \frac{d\mathcal{T}_u(\boldsymbol{M}_0)}{du} = \mathcal{T}_0' \circ \mathcal{T}_u(\boldsymbol{M}_0) = \mathcal{T}_0'(\boldsymbol{M}).$$

Let us denote $W(\boldsymbol{M}) = \mathcal{T}_0'(\boldsymbol{M})$, so $W(\boldsymbol{M})$ is the value at M of a vector field of $\mathbb{R}^n$, which is independent of u.

DEFINITION 3.5.– The vector field $\boldsymbol{M} \in \mathbb{R}^n \longrightarrow W(\boldsymbol{M}) = \mathcal{T}_0'(\boldsymbol{M}) \in \mathbb{R}^n$ is called an ***infinitesimal generator*** of the differentiable Lie group $\{\mathcal{T}_u,\ u \in I \subset \mathbb{R}\}$.

EXAMPLE 3.2.– Let $\begin{cases} x = \lambda\, x_0 \\ y = \dfrac{1}{\lambda}\, y_0 \end{cases}$ be a transformation of $\mathbb{R}^2$, then $\begin{cases} \dfrac{dx}{d\lambda} = x_0 \\ \dfrac{dy}{d\lambda} = -\dfrac{y_0}{\lambda^2} \end{cases}$ and for $\lambda = 1$ (and not for $\lambda = 0$) $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$, consequently, $\begin{cases} \dfrac{dx}{d\lambda} = x \\ \dfrac{dx}{d\lambda} = -y \end{cases}$ is the system of

differential equations associated with the transformation. Let us note that the Lie group is not associated with $\lambda = 0$. Indeed, by writing $\lambda = e^u$, we obtain $\begin{cases} x = e^u\, x_0 \\ y = e^{-u}\, y_0 \end{cases}$, which is a Lie group, and $\begin{cases} \dfrac{dx}{du} = e^u\, x_0 = x \\ \dfrac{dy}{du} = -e^{-u}\, y_0 = -y \end{cases}$.

3.3. Invariant integral under a Lie group

On the one hand, let $\boldsymbol{M} \in \mathbb{R}^n \longrightarrow \boldsymbol{V}^\star(\boldsymbol{M}) \in \mathbb{R}^{n\star}$ be a field of linear forms. Any curve $(\mathcal{C})$, with extremities A and B and also denoted $\widetilde{AB}$, is associated with the curvilinear integral

$$a = \int_{(\mathcal{C})} \boldsymbol{V}^\star(\boldsymbol{M})\, d\boldsymbol{M}. \qquad [3.5]$$

Expression [3.5] defines a functional of the curve $(\mathcal{C})$ that is denoted by

$$a = \mathcal{G}(\mathcal{C}).$$

On the other hand, let $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$ be a differentiable Lie group defined by the *infinitesimal generator* $\boldsymbol{M} \in \mathbb{R}^n \longrightarrow \boldsymbol{W} = \psi(\boldsymbol{M}) \in \mathbb{R}^n$. Here, $(\mathcal{C}_u) = \mathcal{T}_u(\mathcal{C})$ is associated with each value u in the interval J. We obtain the image of $(\mathcal{C})$ by $\mathcal{T}_u$ such that $\mathcal{T}_u(\boldsymbol{M})$ belonging to $(\mathcal{C}_u)$ is associated with each point M in $(\mathcal{C})$

$$t \in I \subset \mathbb{R} \longrightarrow M(t) \in (\mathcal{C}) \longrightarrow \mathcal{T}_u(\boldsymbol{M}(t)) = \boldsymbol{M}_u(t) \in (\mathcal{C}_u)$$

Let us recall that $\boldsymbol{W} = \dfrac{d\boldsymbol{M}}{du}$.

The construction defines a two-parameter family

$$(t, u) \in I \times J \longrightarrow \boldsymbol{M}_u(t) \in (\mathcal{C}_u)\,.$$

DEFINITION 3.6.– The integral $a = \int_{(\mathcal{C})} \boldsymbol{V}^\star(\boldsymbol{M})\, d\boldsymbol{M}$ is said to be invariant under the differentiable Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$ if and only if for any $u \in J$ and for any curve $(\mathcal{C})$, $\mathcal{G}\left(\mathcal{T}_u(\mathcal{C})\right) = \mathcal{G}(\mathcal{C})$.

We also say that the differentiable Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$ keeps a invariant.

A curve that makes a extremal is said to be an extremal curve.

THEOREM 3.3.– **Noether's theorem**.

Given a curvilinear integral $a = \int_{(\mathcal{C})} \boldsymbol{V}^\star(\boldsymbol{M})\ d\boldsymbol{M}$ invariant under the differentiable Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$ of infinitesimal generator $\boldsymbol{W}(\boldsymbol{M})$, then, along the extremals of a, we have the relation $\boldsymbol{V}^\star(\boldsymbol{M})\,\boldsymbol{W}(\boldsymbol{M}) = c_1$, where c_1 is a real constant.

Remarks

a) $\boldsymbol{V}^\star\,\boldsymbol{W}$ scalar defined at each point M of $(\mathcal{C})$ is constant all along an extremal curve; $\boldsymbol{V}^\star\,\boldsymbol{W}$ corresponds to a *law of conservation*, and a is called a ***first integral***.

b) The differentiable Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$ makes it possible to define a deformation at every point of $(\mathcal{C})$: indeed, if we write $\delta\boldsymbol{M} = \boldsymbol{W}(\boldsymbol{M})\,du = (\mathcal{T})'_0(\boldsymbol{M})\,du$, then according to the calculation carried out in section 2.2, this corresponds to a two-parameter family we denoted by ψ.

PROOF.– Consider a variation of the curve $(\mathcal{C})$. In Chapter 2, we saw that

$$\delta a = \int_{(\mathcal{C})} \delta\boldsymbol{V}^\star d\,\boldsymbol{M} - d\boldsymbol{V}^\star\delta\boldsymbol{M} + \boldsymbol{V}(\boldsymbol{B})^\star\delta\boldsymbol{B} - \boldsymbol{V}(\boldsymbol{A})^\star\delta\boldsymbol{A}$$

where A and B are the extremities of $(\mathcal{C})$. This result is independent of the chosen extremities on $(\mathcal{C})$, and if we replace $(\mathcal{C})$ by $(\mathcal{C}')$, which is a curve contained in $(\mathcal{C})$ with extremities A' and B', we obtain the same result. Let us choose:

a) $\delta\boldsymbol{M} = \boldsymbol{W}(\boldsymbol{M})\,du$, where $\boldsymbol{W}(\boldsymbol{M})$ is the infinitesimal generator of the Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$;

b) an extremal curve $(\mathcal{C})$ of a.

On the one hand, since a is invariant by the Lie group, $\delta\boldsymbol{M} = \boldsymbol{W}(\boldsymbol{M})\,du$ implies $\delta a = 0$. The curve $(\mathcal{C})$ is an extremal of a; thus, $\delta\boldsymbol{V}^\star d\,\boldsymbol{M} - d\boldsymbol{V}^\star\delta\boldsymbol{M} = 0$.

On the other hand, we have $\delta\boldsymbol{B} = \boldsymbol{W}(\boldsymbol{B})\,du$ and $\delta\boldsymbol{A} = \boldsymbol{W}(\boldsymbol{A})\,du$. Consequently,

$$\boldsymbol{V}(\boldsymbol{B})^\star\delta\boldsymbol{B}\,du - \boldsymbol{V}(\boldsymbol{A})^\star\delta\boldsymbol{A}\,du = 0 \text{ then } \boldsymbol{V}(\boldsymbol{B})^\star\boldsymbol{W}(\boldsymbol{B}) - \boldsymbol{V}(\boldsymbol{A})^\star\boldsymbol{W}(\boldsymbol{A}) = 0.$$

Points A and B do not play any particular role in the proof, and consequently, $\boldsymbol{V}(\boldsymbol{M})^\star\boldsymbol{W}(\boldsymbol{M})$ is constant along the extremal curve $(\mathcal{C})$. ■

3.4. Further examination of Fermat's principle

The geometric conditions of the study are those used in section 2.5.1. From origin O, the radius vector of the current point M is denoted by r. The refractive index is supposed to depend only on r, and we thus obtain

$$a = \int_{\widetilde{AB}} n(r)\, ds \equiv \int_{r_0}^{r_1} n(r)\, \boldsymbol{T}^{\star} d\boldsymbol{M},$$

where r_0, r_1 correspond to the radius vectors of points A and B, respectively. The integral a is invariant by the rotation of angle θ_0 around origin O. Indeed,

$$ds = \sqrt{1 + r^2\theta'^2}dr, \quad \text{where } \theta = \theta(r)$$

is invariant by the transformation $(r, \theta) \longrightarrow (r, \theta + \theta_0)$, which, in the orthonormal system $O\,\boldsymbol{i}\,\boldsymbol{j}$, is represented by the rotation

$$\begin{cases} x = x_0 \cos\theta - y_0 \sin\theta \\ y = x_0 \sin\theta + y_0 \cos\theta \end{cases}$$

which implies

$$\frac{d\boldsymbol{M}}{d\theta} = \boldsymbol{W}(\boldsymbol{M}) \text{ with } \boldsymbol{W}(\boldsymbol{M}) = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Along the extremals of a, the expression $n(r)\, \boldsymbol{T}^{\star}\, \boldsymbol{W}$ is constant. In Figure 3.1, at point M, v being the angle between the tangent to the curve and the radius vector, we obtain

$$\boldsymbol{T}^{\star}\ \boldsymbol{W} = \|\boldsymbol{T}\|\, \|\boldsymbol{W}\| \cos\left(\frac{\pi}{2} - v\right) = r \sin v.$$

Consequently,

$$n(r)\, r \sin v = c_2.$$

Note: it is possible to get the same result by using the Euler equations obtained in Chapter 2.

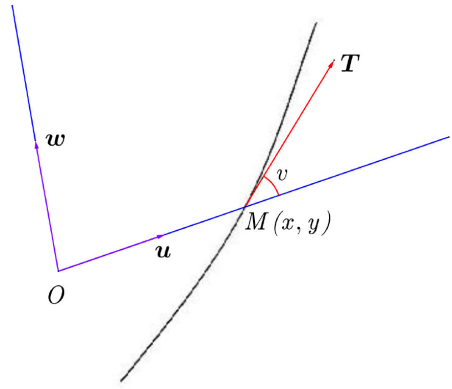


Figure 3.1. *Geometric interpretation of Fermat's principle. The angle made by tangent* $\boldsymbol{T}$ *to the curve at* M *with radius vector* OM *is denoted by* v*. For a color version of this figure, see www.iste.co.uk/gouin/mechanics.zip*

4

The Methods of Analytical Mechanics

4.1. D'Alembert's principle

Although the principle can be studied independently of analytical mechanics methods, we show that it gives the same results, i.e. the Lagrange equations of motion. The reader may learn more details in *Mécanique Générale* by J. Pérès or in *Mechanics* by P. Germain.

4.1.1. *Concept of virtual displacement*

The classical physical space has three dimensions. Any material system is studied with respect to a reference coordinate system (R_0) with origin O. The concept of time, denoted by t, is assumed to be known.

Consider a material system (S) whose position is determined by n parameters denoted by $q_1, \cdots, q_n$ or more simply $\boldsymbol{Q}$ with $\boldsymbol{Q}^\star = [q_1, \cdots, q_n]$. With respect to the coordinate system (R_0), each point M of system (S) is represented by the twice-differentiable vector mapping

$$(\boldsymbol{Q}, t) \in \mathcal{O} \times [t_1, t_2] \subset \mathbb{R}^{n+1} \longrightarrow \boldsymbol{OM}(\boldsymbol{Q}, t) \in \mathbb{R}^3,$$

where $\mathcal{O}$ is an open set in $\mathbb{R}^n$. We deduce

$$d\boldsymbol{OM} = \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}}\, d\boldsymbol{Q} + \frac{\partial \boldsymbol{OM}}{\partial t}\, dt$$

The motion of system $(\mathcal{S})$ with respect to $(\mathcal{R}_0)$ is given by a mapping φ, of class C^2 from $\mathbb{R}$ to $\mathbb{R}^n$

$$t \in I \subset \mathbb{R} \longrightarrow \boldsymbol{Q} = \varphi(t) \in \mathbb{R}^n, \qquad [4.1]$$

where I denotes an open interval of $\mathbb{R}$. The vector $\boldsymbol{V}$ denotes the velocity of point M with respect to (R_0). Thus,

$$\boldsymbol{V} = \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}}\,\dot{\boldsymbol{Q}} + \frac{\partial \boldsymbol{OM}}{\partial t}, \qquad [4.2]$$

where $\dfrac{d\boldsymbol{Q}}{dt}$ is also denoted $\dot{\boldsymbol{Q}}$.

DEFINITION 4.1.– At instant t, the ***virtual displacement*** of M at position $\boldsymbol{Q}$ is the value of differential of $\boldsymbol{OM}(\boldsymbol{Q}, t)$ taken for $dt = 0$.

At fixed time t, it is a differential form and it is common to use the prefix δ to denote a virtual displacement

$$\delta \boldsymbol{OM} = \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}}\,\delta \boldsymbol{Q}.$$

4.1.2. *Concept of constraints*

In mechanical system, the constraints limit the positions and motions. They are of two types.

DEFINITION 4.2.– A ***holonomic constraint*** is a relation of the form

$$\boldsymbol{F}(\boldsymbol{Q}, t) = \boldsymbol{0}, \qquad [4.3]$$

where $\boldsymbol{F}$ is a differentiable mapping from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p < n$.

In coordinates, we write

$$f_j(q_1, \cdots, q_n, t) = 0, \quad j \in \{1, \cdots, p\}.$$

The p relations are assumed to be independent.

DEFINITION 4.3.– A virtual displacement $\delta \boldsymbol{Q}$ is compatible with constraint [4.3] if and only if $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{Q}}\,\delta \boldsymbol{Q} = \boldsymbol{0}$.

Such a constraint reduces the number of independent parameters. Given that the p constraints are independent, it is possible to use the theorem of implicit functions to get $n-p$ independent parameters. The system has $n-p$ *degrees of freedom*. Let us note that a relation of the form

$$f_j(q_1, \cdots, q_n, t) \geq 0$$

is said to be a unilateral relation. The inequality does not reduce the number of independent parameters but only limits the positions of the material system.

DEFINITION 4.4.– A ***non-holonomic constraint*** is a non-integrable relation of the form

$$\boldsymbol{A}(\boldsymbol{Q}, t)\, \dot{\boldsymbol{Q}} + \boldsymbol{B}(\boldsymbol{Q}, t) = \boldsymbol{0} \qquad [4.4]$$

where $\boldsymbol{A}$ is a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^s$ and $\boldsymbol{B}$ is a vector field of $\mathbb{R}^s$ defined at $(\boldsymbol{Q}, t) \in \mathbb{R}^{n+1}$.

$\boldsymbol{A}(\boldsymbol{Q}, t)$ is represented by a matrix with n columns and s rows, and $\boldsymbol{B}(\boldsymbol{Q}, t)$ by a column with s rows. Relation [4.4] can be expressed as

$$a_{i\,1}(q_1, \cdots, q_n, t)\, \dot{q}_1 + \cdots + a_{i\,n}(q_1, \cdots, q_n, t)\, \dot{q}_n + b_i(q_1, \cdots, q_n, t) = 0, \quad i \in \{1, \cdots, s\}.$$

If relation [4.4] is integrable, we are back to the case of relation [4.3]. The matrix $\boldsymbol{A}(\boldsymbol{Q}, t)$ is of rank s with $s < n$.

DEFINITION 4.5.– A virtual displacement $\boldsymbol{Q}$ is compatible with constraint [4.4] if and only if $\boldsymbol{A}(\boldsymbol{Q}, t)\, \delta\boldsymbol{Q} = \boldsymbol{0}$.

4.1.3. *The Lagrange formulae*

Relation [4.2] yields

$$\frac{\partial \boldsymbol{V}}{\partial \dot{\boldsymbol{Q}}} = \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}} \quad \text{hence} \quad \frac{1}{2}\frac{\partial \boldsymbol{V}^\star \boldsymbol{V}}{\partial \dot{\boldsymbol{Q}}} = \boldsymbol{V}^\star \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}}.$$

By deriving with respect to time and taking into account $\dfrac{d}{dt}\left(\dfrac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}}\right) = \dfrac{\partial \boldsymbol{V}}{\partial \dot{\boldsymbol{Q}}}$, we obtain

$$\boldsymbol{a}^\star \frac{\partial \boldsymbol{OM}}{\partial \boldsymbol{Q}} = \frac{d}{dt}\left(\frac{1}{2}\frac{\partial \boldsymbol{V}^\star \boldsymbol{V}}{\partial \dot{\boldsymbol{Q}}}\right) - \frac{1}{2}\frac{\partial \boldsymbol{V}^\star \boldsymbol{V}}{\partial \boldsymbol{Q}} \qquad [4.5]$$

where $\boldsymbol{a} = \dfrac{d\boldsymbol{V}}{dt}$ denotes the acceleration of M. Relation [4.5] is called the Lagrange kinematic formula.

DEFINITION 4.6.– The scalar $m\,\boldsymbol{a}^{\star}\delta\boldsymbol{OM}$ is called ***virtual work of inertial forces*** of point M with mass m.

DEFINITION 4.7.– With motion

$$t \in I \subset \mathbb{R} \longrightarrow \boldsymbol{Q} = \varphi(t) \in \mathcal{O} \subset \mathbb{R}^n$$

is associated the scalar $T\left(\varphi(t), \dot{\varphi}(t), t\right)$, also denoted by $T\left(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t\right)$. T is a mapping from $\mathcal{O} \times \mathcal{O} \times \mathcal{I}$ to $\mathbb{R}$, called the ***kinetic energy*** of system $(\mathcal{S})$ defined by

$$T = \frac{1}{2}\int_{(\mathcal{S})} \|\boldsymbol{V}(M)\|^2\, dm \equiv \frac{1}{2}\int_{(\mathcal{S})} \boldsymbol{V}^{\star}\boldsymbol{V}\, dm. \qquad [4.6]$$

The kinetic energy is usually expressed by using a second-degree polynomial with respect to variables $\dot{q}_1, \cdots, \dot{q}_n$ where polynomial coefficients are functions of parameters $q_1, \cdots, q_n$ and t.

THEOREM 4.1.– The virtual work of the inertial forces of (S) is

$$\int_{(S)} \boldsymbol{a}^{\star}\delta\boldsymbol{OM}\, dm = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{Q}}}\right) - \frac{\partial T}{\partial \boldsymbol{Q}}\right]\delta\boldsymbol{Q},$$

where dm denotes the mass measurement of material system (S).

The proof of the theorem can be quickly deduced from the mass conservation and relations [4.5] and [4.6].

We also assume that system $(\mathcal{S})$ has *a potential energy* represented by the differentiable mapping

$$(t, \boldsymbol{Q}) \in I \times \mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow W(\boldsymbol{Q}, t) \in \mathbb{R},$$

where $\mathcal{O}$ denotes an open set of $\mathbb{R}^n$. The virtual work of forces deriving from the potential energy $W(\boldsymbol{Q}, t)$ is

$$\delta\tau = -\frac{\partial W(\boldsymbol{Q}, t)}{\partial \boldsymbol{Q}}\delta\boldsymbol{Q}.$$

If the forces are not deriving from a potential, it is still assumed that the virtual work of forces is a linear form of the virtual displacement, i.e.

$$\delta\tau = \boldsymbol{R}(\boldsymbol{Q}, t)^{\star}\delta\boldsymbol{Q} \quad \text{with} \quad \boldsymbol{R}(\boldsymbol{Q}, t)^{\star} = \left[r_1(q_1, \cdots, q_n, t), \cdots, r_n(q_1, \cdots, q_n, t)\right].$$

PRINCIPLE 4.1.– *d'Alembert's principle or the principle of virtual work*. There exists a coordinate system (R_0) called a Galilean system and an absolute chronology such that for the coordinate system and the chronology, for any material system, for any virtual displacement compatible with the constraints, the virtual work of forces applied to the system is equal to the virtual work of inertial forces.

Given the holonomic and non-holonomic constraints, the principle can be expressed as

$$\forall\, \delta\boldsymbol{Q} \in \mathbb{R}^n \quad \text{satisfying} \quad \frac{\partial \boldsymbol{F}(\boldsymbol{Q},t)}{\partial \boldsymbol{Q}}\,\delta\boldsymbol{Q} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{Q},t)\,\delta\boldsymbol{Q} = \boldsymbol{0},$$

$$\text{we have} \qquad \boldsymbol{R}(\boldsymbol{Q},t)\,\delta\boldsymbol{Q} - \left[\frac{d}{dt}\left(\frac{\partial T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)}{\partial \dot{\boldsymbol{Q}}}\right) - \frac{\partial T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)}{\partial \boldsymbol{Q}}\right]\delta\boldsymbol{Q} = \boldsymbol{0}.$$

This result is equivalent to the following:

There exist Λ and Ξ, Lagrange multipliers, such that

$$\left[\frac{d}{dt}\left(\frac{\partial T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t).}{\partial \dot{\boldsymbol{Q}}}\right) - \frac{\partial T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)}{\partial \boldsymbol{Q}}\right] = \boldsymbol{R}(\boldsymbol{Q},t)^{\star} + \Lambda\frac{\partial \boldsymbol{F}(\boldsymbol{Q},t).}{\partial \boldsymbol{Q}} + \Xi\boldsymbol{A}(\boldsymbol{Q},t). \qquad [4.7]$$

In coordinate lines, we have

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] = r_i + \sum_{j=1}^{p}\lambda_j\frac{\partial f_j}{\partial q_i} + \sum_{k=1}^{s}\mu_k a_{k,i}, \quad i \in \{1,\cdots,n\}$$

with

$$\Lambda = [\lambda_1,\cdots,\lambda_p], \qquad \Xi = [\mu_1,\cdots,\mu_s] \quad \text{and} \quad p+s<n.$$

These equations are the n Lagrange equations in the general case of holonomic and non-holonomic constraints. In order for the system to have as many equations as the unknowns' number, we must add the $p+s$ constraints,

$$\boldsymbol{F}(\boldsymbol{Q},t) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{Q},t)\,\dot{\boldsymbol{Q}} + \boldsymbol{B}(\boldsymbol{Q},t) = \boldsymbol{0}.$$

4.2. Back to analytical mechanics

Let us consider a system $(\mathcal{S})$ with n degrees of freedom and with holonomic constraints. The position of the system is determined by an appropriate choice of n independent parameters $\boldsymbol{Q}^\star = [q_1, \cdots, q_n]$. In this book, we consider such systems with forces deriving from potential.

The *Lagrangian* associated with system $(\mathcal{S})$ is the differentiable mapping G from $2n+1$ variables $(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) \in \mathbb{R}^{2n+1}$ to $\mathbb{R}$ such that

$$G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = T(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) - W(\boldsymbol{Q}, t). \qquad [4.8]$$

A simpler writing, $G = T - W$, is generally used.

Let us recall the Hamilton principle stated in Chapter 1, section 1.6.3.

We consider a Galilean coordinate system $\mathcal{R}_0$, which is called a fixed coordinate system, for which the positions of system $(\mathcal{S})$ at instants t_0 and t_1 are denoted by $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$, respectively. The motion of system $(\mathcal{S})$ between instants t_0 and t_1 corresponding to positions $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ is given by the mapping

$$t \in I \subset O \longrightarrow Q = \varphi(t) \in \mathbb{R}^n \text{ satisfying } \varphi(t_0) = \boldsymbol{Q}_0 \text{ and } \varphi(t_1) = \boldsymbol{Q}_1$$

and making extremal the Hamilton action

$$a = \int_{t_0}^{t_1} G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)\, dt.$$

We have seen in Chapter 1, section 1.6.3 that the extremals of a satisfy the Euler equations, which are called **Lagrange equations** in mechanics:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial G}{\partial \dot{q}_1}\right) - \dfrac{\partial G}{\partial q_1} = 0, \\ \qquad\qquad \vdots \\ \dfrac{d}{dt}\left(\dfrac{\partial G}{\partial \dot{q}_n}\right) - \dfrac{\partial G}{\partial q_n} = 0, \end{cases} \qquad [4.9]$$

As a function of kinetic energy and potential energy, we obtain

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_1}\right) - \dfrac{\partial T}{\partial q_1} = -\dfrac{\partial W}{\partial q_1}, \\ \qquad\qquad \vdots \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_n}\right) - \dfrac{\partial T}{\partial q_n} = -\dfrac{\partial W}{\partial q_n}. \end{cases}$$

They are the Lagrange equations obtained in [4.7], section 4.1.3 or [1.9] in Chapter 1.

4.3. The vibrating strings

An application of the Hamilton principle is associated with the vibrating strings.

Let us consider an elastic wire that constitutes a string whose ends are fixed at O and A. The string is first stretched out to a length l_0, and ρ denotes the linear mass of the homogeneous wire. In the orthonormal axes Oxy, the unstretched string occupies the position denoted by $M_0(x, 0)$, where $x \in [0, l_0]$. It is hypothesized that the position of the string varies, with respect to its initial position, only by its ordinate. This hypothesis corresponds to the fact that the position of the string varies from the initial position along the line parallel to Oy. At instant t, the string occupies the position $M(x, y)$, where $x \in [0, l_0]$ and $y = f(x, t)$. Consequently, the position of the string is represented by the value $y(x, t)$ (see Figure 4.1).

The kinetic energy of the string is

$$T = \frac{1}{2}\int_O^A \dot{y}^2\, dm = \frac{1}{2}\int_0^{l_0} \rho\, \dot{y}^2\, dx \quad \text{with} \quad \dot{y} = \frac{\partial y}{\partial t}(x, t),$$

where the mass measurement dm is equal to $\rho\, dx$ and ρ is constant. It is supposed that the potential energy of the string is proportional to its stretching relative to the initial unstretched position

$$W = k^2\,(l(t) - l_0) \quad \text{where } l(t) \text{ is the length of the string at instant } t.$$

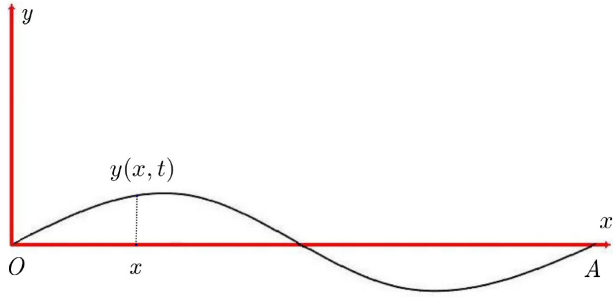


Figure 4.1. *A string connects two fixed points O and A. The position of the string is represented at each instant by the curve $y = f(x, t)$, also simply denoted $y = y(x, t)$. For a color version of the figures in this chapter, see www.iste.co.uk/gouin/mechanics.zip*

The coefficient k^2 is positive and $l(t) = \displaystyle\int_0^{l_0} \sqrt{1+y'^2}\,dx$, where $y' = \dfrac{\partial y}{\partial x}(x,t)$. The value of the Lagrangian is

$$T - W = \frac{1}{2}\int_0^{l_0} \left[\rho\,\dot{y}^2 - 2k^2\left(\sqrt{1+y'^2} - 1\right)\right] dx$$

and between instants t_0 and t_1, the Hamilton action is

$$a = \frac{1}{2}\int_{t_0}^{t_1} \left\{\int_0^{l_0} \left[\rho\,\dot{y}^2 - 2k^2\left(\sqrt{1+y'^2} - 1\right)\right] dx\right\} dt.$$

Because $y(0,t) = y(l_0,t) = 0$, and $y(x,t_0)$ and $y(x,t_1)$ are given, the Hamilton principle implies that whatever δy such that $\delta y(0,t) = \delta y(l_0,t) = 0$ and $\delta y(x,t_0) = \delta y(x,t_1) = 0$, then $\delta a = 0$ where

$$\delta a = \int_{t_0}^{t_1} \left\{\int_0^{l_0} \left(\rho\,\dot{y}\,\delta\dot{y} - k^2 \frac{y'\delta y'}{\sqrt{1+y'^2}}\right) dx\right\} dt.$$

An integration of δa taking into account

$$\dot{y}\,\delta\dot{y} = \frac{d}{dt}\left[\dot{y}\,\delta y\right] - \ddot{y}\,\delta y \quad \text{and} \quad \frac{y'\delta y'}{\sqrt{1+y'^2}} = \frac{d}{dx}\left[\frac{y'\delta y}{\sqrt{1+y'^2}}\right] - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right]\delta y$$

can be written as

$$\delta a = \rho\int_0^{l_0} \left(\left[\dot{y}\,\delta y\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{y}\,\delta y\,dt\right) dx$$
$$- k^2\int_{t_0}^{t_1} \left(\left[\frac{y'\delta y}{\sqrt{1+y'^2}}\right]_0^{l_0} - \int_0^{l_0} \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\delta y\,dx\right) dt.$$

The terms within the square brackets [ ] are zero; therefore,

$$\delta a = \int_{t_0}^{t_1} \left(\int_0^{l_0} \left[-\rho\,\ddot{y} + k^2\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)\right]\delta y\,dx\right) dt.$$

If, for any δy, we get $\delta a = 0$, then we obtain the differential equation

$$-\rho\,\ddot{y} + k^2 \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0. \qquad [4.10]$$

Let us develop the second term in [4.10],

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{y''}{\sqrt{1+y'^2}} - \frac{y'^2 y''}{\sqrt{(1+y'^2)^3}} = \frac{y''}{\sqrt{(1+y'^2)^3}}.$$

When we assume $|y'| \ll 1$. This term can be expanded in the form

$$\frac{y''}{\sqrt{(1+y'^2)^3}} = y''\left(1 - \frac{3}{2}\,y'^2 + \cdots\right).$$

We are in the case of infinitesimal motions of the vibrating string. Therefore, the linearized motion equation of vibrating strings writes

$$\rho\,\ddot{y} - k^2\,y'' = 0. \qquad [4.11]$$

4.3.1. *First study of the solutions of equation [4.11]*

We are searching for solutions in a form that separates space and time

$$y = f(x)\cos\omega t.$$

For any t, we obtain in [4.11],

$$\cos\omega t\left[\,\rho\,\omega^2 f(x) + k^2 f''(x)\,\right] = 0 \quad \Longrightarrow \quad f''(x) + \rho\,\frac{\omega^2}{k^2} f(x) = 0.$$

Then, researching for $f(x)$ in the form $f(x) = e^{rx}$ where r is a constant scalar, we obtain

$$r^2 + \rho\,\frac{\omega^2}{k^2} = 0, \ \text{ or } \ r = \pm\, i\,\alpha \ \text{ with } \ \alpha > 0 \ \text{ and } \ \alpha^2 = \rho\,\frac{\omega^2}{k^2}$$

and consequently,

$$f(x) = A\cos(\alpha\,x) + B\sin(\alpha\,x).$$

The ends of the string are fixed. Thus, $y(0,t) = y(l_0,t) = 0$, and we obtain

$$f(0) = 0 \implies A = 0 \quad \text{and} \quad f(l_0) = 0 \implies B \sin l_0 = 0, \text{ i.e. } \alpha\, l_0 = n\,\pi.$$

If the solutions are not identically zero (corresponding to the unstretched state of the string), we have an infinite number of values corresponding to

$$\alpha_n = n\,\frac{\pi}{l_0} \quad \text{or} \quad \omega_n = \frac{k}{l_0\sqrt{\rho}}\,n\,\pi.$$

The different values of n correspond to all the *harmonics* of the same vibrating string's note. The note is associated with k and l_0. The value of B is arbitrary and characterizes the *note's intensity*. The general solution of equation [4.11] is

$$y(x,t) = \sum_{p=1}^{p=n} B_p \sin\left(p\,\frac{\pi}{l_0}\,x\right)\cos\left(p\,\frac{\pi\,k}{l_0\sqrt{\rho}}\,t\right).$$

4.3.2. *Second study of the solutions of equation [4.11]*

We search for the solutions of the vibrating strings by using the Cauchy problem and the method of the waves' propagation. Equation [4.11] takes the form of a partial differential equation,

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0, \qquad [4.12]$$

where $v^2 = \dfrac{\rho}{k^2}$. The unknown function y depends on x and t. To find the general solution of [4.12], let us change variables:

$$\begin{cases} \zeta = x + vt \\ \eta = x - vt \end{cases} \iff \begin{cases} x = \dfrac{\zeta + \eta}{2} \\ t = \dfrac{\zeta - \eta}{2v} \end{cases}.$$

Since we have the relations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = v\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \eta}\right),$$

we obtain

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \zeta^2} + 2\frac{\partial^2}{\partial \zeta\,\partial \eta} + \frac{\partial^2}{\partial \eta^2} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial \zeta^2} - 2\frac{\partial^2}{\partial \zeta\,\partial \eta} + \frac{\partial^2}{\partial \eta^2}\right),$$

and

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = -4\frac{\partial^2 y}{\partial \zeta \partial \eta}.$$

Using new variables ζ and η, the partial differential equation [4.12] becomes

$$\frac{\partial^2 y}{\partial \zeta \partial \eta} = 0. \qquad [4.13]$$

The general solution of [4.13] is

$$y(\zeta, \eta) = f(\zeta) + g(\eta) \quad \Longrightarrow \quad y(x,t) = f(x+vt) + g(x-vt),$$

where f and g are two arbitrary, two-times continuously differentiable functions. At point x and at instant t, the function $g(x-vt)$ takes the value that it would take at the point $x-vt$ at instant 0. The value of $g(x-vt)$ is the translation of $g(x)$ by translation $v\,t$. The function $y(x,t) = g(x-vt)$ represents *a wave*, which propagates towards the *right* with velocity v. The function $y(x,t) = f(x+vt)$ represents a wave which is propagated towards the *left* with velocity v. Any wave of the vibrating string is made up of the superimposition of these two waves.

PROPOSITION 4.1.– ***The Cauchy problem***. The Cauchy problem is as follows: we search for a solution of the vibrating strings' equation, which corresponds to given initial conditions.

The form of the string at the initial instant is: $y_0(x) = y(x,0)$. The velocities along the string at the initial instant are $\dot{y}_0(x) = \dfrac{\partial y}{\partial t}(x,0)$. Then,

$$y_0(x) = f(x) + g(x) \text{ and } \dot{y}_0(x) = v\left(f'(x) - g'(x)\right). \qquad [4.14]$$

Second equation [4.14] implies

$$\int_0^x \dot{y}_0(\zeta)\, d\zeta = v\left(f(x) - g(x)\right) + c \quad \text{where } c \text{ is a constant.}$$

Hence,

$$\begin{cases} f(x) = \dfrac{1}{2}\left(y_0(x) + \dfrac{1}{v}\displaystyle\int_0^x \dot{y}_0(\zeta)\, d\zeta - \dfrac{c}{v}\right) \\ g(x) = \dfrac{1}{2}\left(y_0(x) - \dfrac{1}{v}\displaystyle\int_0^x \dot{y}_0(\zeta)\, d\zeta + \dfrac{c}{v}\right) \end{cases}.$$

The constant c in f is simplified with the constant c in g, and we obtain the solution of the Cauchy problem

$$y(x,t) = \frac{1}{2}\left[y_0(x+vt) + y_0(x-vt)\right] + \frac{1}{2v}\int_{x-vt}^{x+vt} \dot{y}_0(\zeta)\, d\zeta. \qquad [4.15]$$

EXAMPLE 4.1.– ***Vibrating string with one fixed end.***

The solutions are given by [4.15], where $y(x,t)$ is defined from $[0,\ell_0]\times[t_0,+\infty]$ to $\mathbb{R}$. Indeed, any solution can be extended in a unique manner at $t\in]-\infty,+\infty[$. For a fixed end point O and for any t, we have $y(0,t)=0$. Thus, $f(vt)+g(-vt)=0$, and for any s, $g(s)=-f(s)$. Finally,

$$y(x,t) = f(x+vt) - f(x-vt)$$

where f is an arbitrary function. At x, $y(x,t)$ is an odd function and is the superimposition of the two waves $y_1(x,t)$ and $y_2(x,t)$ propagating in opposite directions and satisfying, whatever t, to the relation

$$y_1(-x,t) = -y_2(x,t).$$

There is a reflection of waves at point O with a change of sign for the amplitude of the wave's oscillation.

EXAMPLE 4.2.– ***Vibrating string with two fixed ends.***

The solutions are given by [4.15], where $y(x,t)$ is defined from $[0,\ell_0]\times[t_0,+\infty]$ to $\mathbb{R}$. We obtain

$$\begin{cases} f(vt)+g(-vt)=0 \\ f(l_0+vt)+g(l_0-vt)=0 \end{cases} \quad\Longrightarrow\quad f(l_0+s)=f(-l_0+s).$$

The function f is periodic with period $2\,l_0$ and, as in the previous example, can be extended in $\overline{y}$ such that

$$\begin{cases} \overline{y}(-x,t) = -\overline{y}(x,t) \\ \overline{y}(x+2l_0,t) = \overline{y}(x,t) \end{cases} \quad \text{and} \quad \overline{y}(x,\frac{v\,t+2l_0}{v}) = \overline{y}(x,t).$$

The relations imply $\overline{y}(0,t)=\overline{y}(l_0,t)=0$.

4.4. Homogeneous Lagrangian. Expression in space time

The motion of system $(\mathcal{S})$ relative to the coordinate system $(\mathcal{R}_0)$ is represented by the mapping [4.1]. The motion can also be expressed in the form of parametric representation by writing $t = g(\tau)$, where g is continuously differentiable and a bijective function from an interval I to an interval J of $\mathbb{R}$. Thus,

$$\begin{cases} \boldsymbol{Q} = \psi(\tau), \\ t = g(\tau) \end{cases} \qquad [4.16]$$

The parametric representation [4.16] is the representation of the trajectory of system $(\mathcal{S})$ in space time. Writing

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix},$$

then $\dfrac{d\boldsymbol{X}}{d\tau}$ is denoted by $\boldsymbol{X}' = \begin{bmatrix} \boldsymbol{Q}' \\ t' \end{bmatrix}$ representing the column of derivatives with respect to parameter τ. By noting that $\dot{\boldsymbol{Q}} = \dfrac{\boldsymbol{Q}'}{t'}$, the Hamilton action is written as

$$a = \int_{t_0}^{t_1} G(\boldsymbol{Q}, \frac{\boldsymbol{Q}'}{t'}, t)\, dt = \int_{\tau_0}^{\tau_1} G(\boldsymbol{Q}, \frac{\boldsymbol{Q}'}{t'}, t)\, t'\, d\tau \quad \text{where} \quad \begin{cases} \tau_0 = g^{-1}(t_0) \\ \tau_1 = g^{-1}(t_1) \end{cases}.$$

We write $L(\boldsymbol{X}, \boldsymbol{X}') = G(\boldsymbol{Q}, \dfrac{\boldsymbol{Q}'}{t'}, t)\, t'$. Thus, L is called the homogeneous Lagrangian of system $(\mathcal{S})$. Since L is a homogeneous function of $\boldsymbol{X}'$, of degree 1, the Euler identity yields

$$L = \frac{\partial L(\boldsymbol{X}, \boldsymbol{X}')}{\partial \boldsymbol{X}'}\, \boldsymbol{X}',$$

or

$$L(\boldsymbol{X}, \boldsymbol{X}') = \frac{\partial L}{\partial q_1'}\, q_1' + \cdots + \frac{\partial L}{\partial q_n'}\, q_n' + \frac{\partial L}{\partial t'}\, t'.$$

We obtain

$$a = \int_{\tau_0}^{\tau_1} \frac{\partial L(\boldsymbol{X}, \boldsymbol{X}')}{\partial \boldsymbol{X}'}\, \boldsymbol{X}' d\tau.$$

Let us write $d\boldsymbol{X} = \boldsymbol{X}' d\tau$ and $\boldsymbol{Y}^\star = \dfrac{\partial L(\boldsymbol{X}, \boldsymbol{X}')}{\partial \boldsymbol{X}'}$, with

$$\frac{\partial L}{\partial \boldsymbol{X}'} = \left[\frac{\partial L}{\partial X_1'}, \cdots, \frac{\partial L}{\partial X_{n+1}'} \right],$$

then,

$$a = \int_{\tau_0}^{\tau_1} \boldsymbol{Y}^\star \boldsymbol{X}' \, d\tau = \int_{\boldsymbol{X}_0}^{\boldsymbol{X}_1} \boldsymbol{Y}^\star \, d\boldsymbol{X} \quad \text{with} \quad \boldsymbol{X}(\tau_0) = \boldsymbol{X}_0 \text{ and } \boldsymbol{X}(\tau_1) = \boldsymbol{X}_1.$$

The vector $\boldsymbol{Y}$ is called the *energy momentum vector* of system $(\mathcal{S})$. The Hamilton action is the integral or circulation in space time of the energy momentum vector. Let us write

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{P} \\ h \end{bmatrix} \quad \text{and} \quad d\boldsymbol{X} = \begin{bmatrix} d\boldsymbol{Q} \\ dt \end{bmatrix},$$

thus,

$$\boldsymbol{Y}^\star d\boldsymbol{X} = \boldsymbol{P}^\star d\boldsymbol{Q} + h \, dt.$$

The covector $\boldsymbol{P}^\star$ is associated with the momentum, and the scalar h is associated with the energy. Calculating $\boldsymbol{P}$ and h,

$$\boldsymbol{Y}^\star = [\boldsymbol{P}^\star, h] = \frac{\partial L(\boldsymbol{X}, \boldsymbol{X}')}{\partial \boldsymbol{X}'} = \left[\frac{\partial L}{\partial \boldsymbol{Q}'}, \frac{\partial L}{\partial t'} \right].$$

Since $L(\boldsymbol{X}, \boldsymbol{X}') = G(\boldsymbol{Q}, \dfrac{\boldsymbol{Q}'}{t'}, t)\, t'$ and $\dot{\boldsymbol{Q}} = \dfrac{\boldsymbol{Q}'}{t'}$, we obtain $\dfrac{\partial \dot{\boldsymbol{Q}}}{\partial \boldsymbol{Q}'} = \dfrac{1}{t'}$, and consequently,

$$\boldsymbol{P}^\star = \frac{\partial L}{\partial \boldsymbol{Q}'} = \frac{\partial G t'}{\partial \dot{\boldsymbol{Q}}} \frac{\partial \dot{\boldsymbol{Q}}}{\partial \boldsymbol{Q}'} = t' \frac{\partial G}{\partial \dot{\boldsymbol{Q}}} \frac{\partial \dot{\boldsymbol{Q}}}{\partial \boldsymbol{Q}'} = \frac{\partial G}{\partial \dot{\boldsymbol{Q}}},$$

$$h = \frac{\partial L}{\partial t'} = G - t' \frac{\partial G}{\partial \dot{\boldsymbol{Q}}} \frac{\boldsymbol{Q}'}{t'^2} = G - \frac{\partial G}{\partial \dot{\boldsymbol{Q}}} \dot{\boldsymbol{Q}},$$

hence, we obtain the system

$$\begin{cases} \boldsymbol{P} = \left(\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}} \right)^\star = \left(\dfrac{\partial T}{\partial \dot{\boldsymbol{Q}}} \right)^\star, \\ h = G - \dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}} \dot{\boldsymbol{Q}}. \end{cases} \qquad [4.17]$$

REMARK 4.1.– The kinetic energy is generally a second-degree polynomial with respect to the derivatives of the parameters $\dot{q}_1, \cdots, \dot{q}_n$ ($\dot{\boldsymbol{Q}}^\star = [\dot{q}_1, \cdots, \dot{q}_n]$). We write

$$T = T_2 + T_1 + T_0$$

where:

T_2 is homogeneous of degree 2 in $\dot{\boldsymbol{Q}}$;

T_1 is homogeneous of degree 1 in $\dot{\boldsymbol{Q}}$;

T_0 is independent of $\dot{\boldsymbol{Q}}$.

Thus, according to the Euler identity,

$$\frac{\partial T}{\partial \dot{\boldsymbol{Q}}}\,\dot{\boldsymbol{Q}} = 2\,T_2 + T_1,$$

and we obtain $h = T - W - 2\,T_2 - T_1$, or

$$h = -(T_2 - T_0 + W).$$

For example, the kinetic energy of a material point can be written in orthonormal axis as: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; hence, $\boldsymbol{P}^\star = [\,m\dot{x}, m\dot{y}, m\dot{z}\,]$ and $h = -(T + W)$.

As the kinetic energy is a second-degree polynomial relatively to $\dot{q}_i, (i \in \{1, \ldots, n\})$, term $\dfrac{\partial T}{\partial \dot{\boldsymbol{Q}}}$ is a linear form whose n components $\dfrac{\partial T}{\partial \dot{q}_i}$ are first-degree polynomials in $\dot{q}_i$. This means that $\boldsymbol{P}^\star = \dfrac{\partial T}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)$ is a system of n linear equations with respect to $[\,\dot{q}_1, \cdots, \dot{q}_n\,]$ whose coefficients depend on $\boldsymbol{Q}^\star = [\,q_1, \cdots, q_n\,]$. Moreover, $\boldsymbol{P}^\star = [\,p_1, \cdots, p_n\,]$ are the components of the momentum vector.

Solving the system $\boldsymbol{P}^\star = \dfrac{\partial T}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)$ makes it possible to write

$$\dot{\boldsymbol{Q}} = \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \begin{bmatrix} f_1(\boldsymbol{Q}, \boldsymbol{P}, t) \\ \vdots \\ f_n(\boldsymbol{Q}, \boldsymbol{P}, t) \end{bmatrix}, \quad \text{that is} \quad \dot{\boldsymbol{Q}} = F(\boldsymbol{Q}, \boldsymbol{P}, t).$$

The resolution is only possible if the system is a Cramer system. This is assumed in the next sections. The term $\dot{\boldsymbol{Q}}$ is expressed as a function of $\boldsymbol{P}$, $\boldsymbol{Q}$ and t. Consequently,

$$h = G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) - \frac{\partial G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)}{\partial \dot{\boldsymbol{Q}}}\, \dot{\boldsymbol{Q}}$$

and we obtain an expression of the form

$$h + H\left(\boldsymbol{Q}, \boldsymbol{P}, t\right) = 0,$$

with

$$H = \boldsymbol{P}^{\star}\dot{\boldsymbol{Q}} - G \qquad [4.18]$$

where $\dot{\boldsymbol{Q}}$ is expressed as a function of $\boldsymbol{Q}$, $\boldsymbol{P}$ and t. The new term $H(\boldsymbol{Q}, \boldsymbol{P}, t)$ is called the *Hamiltonian of the mechanical system.*

If T is a homogeneous second-degree term of $\dot{q}_1, \cdots, \dot{q}_n$, the value of H represents the total energy of system $(\mathcal{S})$.

4.5. The Hamilton equations

4.5.1. *First method using Lagrange equations*

The Lagrangian G of the system $(\mathcal{S})$ is defined by

$$G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = T(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) - W(\boldsymbol{Q}, t)$$

and the momentum $\boldsymbol{P}^{\star}$ is defined by [4.17]; let us change the variables as

$$\left(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t\right) \longrightarrow (\boldsymbol{Q}, \boldsymbol{P}, t)\,.$$

In order to simplify the notation, we write $G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)$ in the simplified form G and $H(\boldsymbol{Q}, \boldsymbol{P}, t)$ in the simplified form H. We can deduce, from [4.18] regarding the Hamiltonian H,

$$\begin{aligned} dH &= d\left(\boldsymbol{P}^{\star}\dot{\boldsymbol{Q}} - G\right) \\ &= d\boldsymbol{P}^{\star}\dot{\boldsymbol{Q}} + \boldsymbol{P}^{\star}d\dot{\boldsymbol{Q}} - \frac{\partial G}{\partial \boldsymbol{Q}}\, d\boldsymbol{Q} - \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\, d\dot{\boldsymbol{Q}} - \frac{\partial G}{\partial t}\, dt, \end{aligned}$$

therefore,

$$\boldsymbol{P}^\star = \frac{\partial G}{\partial \dot{\boldsymbol{Q}}} \quad \Longrightarrow \quad dH = -\frac{\partial G}{\partial \boldsymbol{Q}}\, d\boldsymbol{Q} + \dot{\boldsymbol{Q}}^\star d\boldsymbol{P} - \frac{\partial G}{\partial t}\, dt.$$

By construction, the Hamiltonian H is a function of $2n+1$ variables $q_1, \cdots, q_n$, $p_1, \cdots, p_n, t$ and, by identification of the components of dH, we obtain

$$\begin{cases} \dfrac{\partial H}{\partial \boldsymbol{P}} = \dot{\boldsymbol{Q}}^\star \\ \dfrac{\partial H}{\partial \boldsymbol{Q}} = -\dfrac{\partial G}{\partial \boldsymbol{Q}} \\ \dfrac{\partial H}{\partial t} = -\dfrac{\partial G}{\partial t} \end{cases}$$

PROPOSITION 4.2.– ***The Hamilton equations of motion***

The Hamilton principle yields the Lagrange equations [4.9] for the motion of system $(\mathcal{S})$. The momentum is defined by [4.17],

$$\boldsymbol{P}^\star = \frac{\partial G}{\partial \dot{\boldsymbol{Q}}} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{Q}}\right) - \frac{\partial G}{\partial Q} = 0 \quad \Longrightarrow \quad \dot{\boldsymbol{P}}^\star = \frac{\partial G}{\partial Q}.$$

Hence, *the system of Hamilton equations* is

$$\begin{cases} \dfrac{d\boldsymbol{Q}}{dt} = \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^\star \\ \dfrac{d\boldsymbol{P}}{dt} = -\left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^\star \end{cases} \qquad [4.19]$$

– The Hamilton equations are associated with the change of variables, which replaces the n second-order differential equations of Lagrange by $2n$ first-order differential equations.

– Since

$$\frac{dH}{dt} = \frac{\partial H}{\partial \boldsymbol{Q}}\, \dot{\boldsymbol{Q}} + \frac{\partial H}{\partial \boldsymbol{P}}\, \dot{\boldsymbol{P}} + \frac{\partial H}{\partial t}$$

and by taking into account the Hamilton equations [4.19], we obtain

$$\frac{\partial H}{\partial \boldsymbol{P}}\, \dot{\boldsymbol{P}} + \frac{\partial H}{\partial \boldsymbol{Q}}\, \dot{\boldsymbol{Q}} = \dot{\boldsymbol{Q}}^\star \dot{\boldsymbol{P}} - \dot{\boldsymbol{P}}^\star \dot{\boldsymbol{Q}} = 0,$$

and for any motion of the system $(\mathcal{S})$,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

When H or G are explicitly independent of time,

$$\frac{\partial H}{\partial t} = 0, \text{ then } \frac{dH}{dt} = 0 \quad \text{hence} \quad H(\boldsymbol{Q}, \boldsymbol{P}, t) = H_0,$$

where H_0 is a constant that only depends on the initial conditions of the motion of system $(\mathcal{S})$.

– When H or G are explicitly independent of time and the kinetic energy is a second-degree polynomial with respect to the derivatives of the parameters,

$$T_2 - T_0 + W = H_0.$$

Indeed, if G is independent of t, the homogeneous Lagrangian L defined in section 4.2 is explicitly independent of t, and for any real u,

$$L(\boldsymbol{Q}, t, \boldsymbol{Q}', t') = L(\boldsymbol{Q}, t + u, \boldsymbol{Q}', t').$$

Since the group of translations $\{\mathcal{T}_u,\ u \in I \subset \mathbb{R}\}$ associated with the transformation

$$\mathcal{T}_u \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} \\ t + u \end{bmatrix}$$

has the infinitesimal operator $\boldsymbol{W} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}$, and since $a = \displaystyle\int_{\boldsymbol{X}_0}^{\boldsymbol{X}_1} \boldsymbol{Y}^\star \, d\boldsymbol{X}$ is invariant by the Lie group $\{\mathcal{T}_u,\ u \in J \subset \mathbb{R}\}$, according to the Noether theorem, along the extremal curve representing the motion of system $(\mathcal{S})$, we have

$$\boldsymbol{Y}^\star \boldsymbol{W} = h_1.$$

However, $\boldsymbol{Y}^\star = [\, \boldsymbol{P}^\star, h \,]$; then, $h = h_1$, where h_1, value of $-H$, is constant.

4.5.2. *Second method using the Hamilton principle*

The Hamilton principle leads to search for curves $(\mathcal{C})$ in space time that make extremal the integral

$$a = \int_{(\mathcal{C})} L(\boldsymbol{X}, \boldsymbol{X}') \, d\tau.$$

We previously write

$$a = \int_{(\mathcal{C})} \boldsymbol{Y}^\star \, d\boldsymbol{X} = \int_{(\mathcal{C})} \boldsymbol{P}^\star \, d\boldsymbol{Q} + h \, dt,$$

with $h + H(\boldsymbol{Q}, \boldsymbol{P}, t) = 0$, which represents a constraint between $\boldsymbol{Q}, \boldsymbol{P}, t$ and h. Due to $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix}$ and $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{P} \\ h \end{bmatrix}$, the constraint is written $F(\boldsymbol{X}, \boldsymbol{Y}) = 0$. We will now find a curve $(\mathcal{C})$ and a vector field $\boldsymbol{Y}$ satisfying $F(\boldsymbol{X}, \boldsymbol{Y}) = 0$, which make extremal the curvilinear integral $\int_{(\mathcal{C})} \boldsymbol{Y}^\star \, d\boldsymbol{X}$.

The problem was solved in Chapter 3 as follows:

Find a curve $\widetilde{AB}$ with the given ends A and B and a vector field $\boldsymbol{V}$ defined along $\widetilde{AB}$, subject to the constraint $F(\boldsymbol{M}, \boldsymbol{V}) = 0$, where M is the current point on $\widetilde{AB}$ and such that $\int_{\widetilde{AB}} \boldsymbol{V}^\star \, d\boldsymbol{M}$ is an extremum.

The solution is associated with the construction of a distributed Lagrange multiplier corresponding to the constraint written as

$$\exists \, s \in I \subset \mathbb{R} \longrightarrow (\boldsymbol{M}(s), \boldsymbol{V}(s), \lambda(s)) \text{ such that } \forall \, (\delta\boldsymbol{M}, \delta\boldsymbol{V}, \delta\lambda),$$
$$\delta \int_{\widetilde{AB}} \boldsymbol{V}^\star \, d\boldsymbol{M} - \lambda(s) \, F(\boldsymbol{M}, \boldsymbol{V}) \, ds = 0.$$

We deduce

$$\forall \, (\delta\boldsymbol{M}, \delta\boldsymbol{V}), \int_{\widetilde{AB}} \delta\boldsymbol{V}^\star \, d\,\boldsymbol{M} - d\boldsymbol{V}^\star \, \delta\boldsymbol{M} - \lambda(s) \left[\frac{\partial F}{\partial \boldsymbol{V}} \, \delta\,\boldsymbol{V} + \frac{\partial F}{\partial \boldsymbol{M}} \, \delta\boldsymbol{M} \right] ds = 0,$$

with $F(\boldsymbol{M}, \boldsymbol{V}) = 0$, and we obtain

$$d\,\boldsymbol{M} = \lambda(s) \left(\frac{\partial F}{\partial \boldsymbol{V}} \right)^\star ds \quad \text{and} \quad d\boldsymbol{V} = -\lambda(s) \left(\frac{\partial F}{\partial \boldsymbol{M}} \right)^\star ds.$$

For notations of the homogeneous Lagrangian, we get

$$d\,\boldsymbol{X} = \lambda(\tau) \left(\frac{\partial F}{\partial \boldsymbol{Y}} \right)^\star d\tau \quad \text{and} \quad d\boldsymbol{Y} = -\lambda(\tau) \left(\frac{\partial F}{\partial \boldsymbol{X}} \right)^\star d\tau. \qquad [4.20]$$

Moreover, $F(\boldsymbol{X},\boldsymbol{Y}) = h + H(\boldsymbol{Q},\boldsymbol{P},t)$ with $\boldsymbol{X} = \begin{bmatrix}\boldsymbol{Q}\\ t\end{bmatrix}$ and $\boldsymbol{Y} = \begin{bmatrix}\boldsymbol{P}\\ h\end{bmatrix}$ and [4.20] is equivalent to

$$\begin{cases} d\boldsymbol{Q} = \mu \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} \\ dt = \mu \end{cases} \quad \text{and} \quad \begin{cases} d\boldsymbol{P} = -\mu \left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} \\ dh = -\mu \dfrac{\partial H}{\partial t} \end{cases} \quad \text{with } \mu = \lambda\, ds.$$

By eliminating μ and by taking the constraint into account, we obtain the system

$$\begin{cases} \dfrac{d\boldsymbol{Q}}{dt} = \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} \\ \dfrac{d\boldsymbol{P}}{dt} = -\left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} \\ \dfrac{dh}{dt} + \dfrac{\partial H}{\partial t} = 0 \end{cases} \quad \text{with} \quad h + H(\boldsymbol{Q},\boldsymbol{P},t) = 0. \qquad [4.21]$$

Let us note that $\dfrac{dh}{dt} + \dfrac{\partial H}{\partial t} = 0$ results from the first two equations and the constraint $h + H(\boldsymbol{Q},\boldsymbol{P},t) = 0$. Conversely, if we note that $H = T_2 - T_0 + W$ and that W is defined up to a constant, the constraint $h + H(\boldsymbol{Q},\boldsymbol{P},t) = 0$ results from three equations of system [4.21]; it is similar for H. In the same way,

$$dF(\boldsymbol{X},\boldsymbol{Y}) = \frac{\partial F}{\partial \boldsymbol{X}}\, d\boldsymbol{X} + \frac{\partial F}{\partial \boldsymbol{Y}}\, d\boldsymbol{Y} = \mu \left(\frac{\partial F}{\partial \boldsymbol{X}} \left(\frac{\partial F}{\partial \boldsymbol{Y}}\right)^{\star} - \frac{\partial F}{\partial \boldsymbol{Y}} \left(\frac{\partial F}{\partial \boldsymbol{X}}\right)^{\star}\right) = 0,$$

therefore, $F(\boldsymbol{X},\boldsymbol{Y}) = c_3$, scalar c_3 being constant that can be chosen to be zero.

CONCLUSION 4.1.– Let us summarize the previous steps. For a system $(\mathcal{S})$ with n independent parameters $\boldsymbol{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$:

a) we calculate $T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)$ and $W(\boldsymbol{Q},t)$, and we obtain $G(\boldsymbol{Q},\dot{\boldsymbol{Q}},t) = T(\boldsymbol{Q},\dot{\boldsymbol{Q}},t) - W(\boldsymbol{Q},t)$;

b) we write $\boldsymbol{P}^{\star} = \dfrac{\partial G(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)}{\partial \dot{\boldsymbol{Q}}}$, value of the momentum vector; the n terms $p_1,\cdots,p_n$ are the conjugate variables of $q_1,\cdots,q_n$;

c) we deduce $\dot{\boldsymbol{Q}}$ as a function of $\boldsymbol{P}$ and $\boldsymbol{Q}$;

d) we calculate $-H(\boldsymbol{Q},\boldsymbol{P},t) = G(\boldsymbol{Q},\dot{\boldsymbol{Q}},t) - \dfrac{\partial G(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)}{\partial \dot{\boldsymbol{Q}}}\,\dot{\boldsymbol{Q}}$, with $\dot{\boldsymbol{Q}}$ being expressed as a function of $\boldsymbol{Q}$ and $\boldsymbol{P}$.

If $T - W$ is explicitly independent of time, we obtain the cases $H = T + W$ or $H = T_2 - T_0 + W$. Knowing the Hamiltonian H, we can deduce the $2n$ independent first-order Hamilton equations.

EXAMPLE 4.3.– For a material point and relatively to an orthonormal coordinate system, the kinetic energy and potential energy are

$$T = \frac{1}{2}\,m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \quad \text{and} \quad W = W(x,y,z),$$

thus, $G = T - W$ and $\begin{cases} p_x = m\dot{x} \\ p_y = m\dot{y} \\ p_z = m\dot{z} \end{cases}$, hence

$$H = \frac{1}{2}\,m\left(p_x^2 + p_y^2 + p_z^2\right) + W(x,y,z),$$

and the Hamilton equations are

$$\begin{cases} \dot{x} = \dfrac{p_x}{m} \\ \dot{y} = \dfrac{p_y}{m} \\ \dot{z} = \dfrac{p_z}{m} \end{cases} \quad \text{and} \quad \begin{cases} \dot{p}_x = -\dfrac{\partial W}{\partial x} \\ \dot{p}_y = -\dfrac{\partial W}{\partial x} \\ \dot{p}_z = -\dfrac{\partial W}{\partial x} \end{cases}$$

We also have the Painlevé theorem,

$$\frac{\partial H}{\partial t} = 0 \quad \Longrightarrow \quad H = H_0 \text{ or } T + W = H_0.$$

4.6. First integral by using the Noether theorem

4.6.1. *Secondary parameters*

Let $G = G\,(\,q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$ denote the Lagrangian of system $(\mathcal{S})$ with respect to a Galilean frame (R_0).

DEFINITION 4.8.– The parameter q_i, where $i \in \{1, \cdots, n\}$, is said secondary parameter if and only if $\dfrac{\partial G}{\partial q_i} = 0$, i.e. $G\left(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t\right)$ is independent of the variable q_i.

In section 4.5.1 we saw that $\dfrac{\partial H(\boldsymbol{Q}, \boldsymbol{P}, t)}{\partial \boldsymbol{Q}} = -\dfrac{\partial G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)}{\partial \boldsymbol{Q}}$. So, the definition is equivalent to the Hamiltonian independent of q_i. The i-th Lagrange equation writes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial G}{\partial q_i} = 0 \quad \text{hence} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = 0.$$

There exists a constant c_i such that

$$\frac{\partial T}{\partial \dot{q}_i} = c_i \quad \Longleftrightarrow \quad p_i = c_i.$$

The conjugate variable of the secondary parameter is constant.

4.6.2. *Back to the Noether theorem*

Parameter q_1 is a secondary parameter. Let $\{\mathcal{T}_u,\, u \in J \subset \mathbb{R}\}$ be the group of translations

$$[q_1, q_2, \cdots, q_n, t] \longrightarrow [q_1 + u, q_2 \cdots, q_n, t]\,.$$

The transformation $\mathcal{T}_u$ is associated with the infinitesimal operator

$$\boldsymbol{W} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For the motions of system $(\mathcal{S})$, we get $\boldsymbol{Y}^{\star}\boldsymbol{W} = c_1$, then $p_1 = c_1$.

If t is a secondary parameter, $\dfrac{\partial H}{\partial t} = 0$ and therefore h is constant.

THEOREM 4.2.– ***The Painlevé theorem***. If constraints are independent of time, which is a secondary parameter,

for $T = T_2$, then $H = T + W$ is constant.

For $T = T_2 + T_1 + T_0$, then $H = T_2 - T_0 + W$ is constant.

EXAMPLE 4.4.– Consider a system with n independent parameters $q_1, \cdots, q_n$ such that t and q_1 are secondary parameters and the possible constraints are independent of time. We consider the transformation

$$\mathcal{T}(u,v): \ [q_1, q_2, \cdots, q_n, t] \longrightarrow [q_1 + u, \ q_2, \cdots, q_n, t + v] \, .$$

When $T = T_2 + T_1\,\dot{q}_1 + T_0\,\dot{q}_1^2$, the transformation makes G invariant. From the invariance relative to $\{\mathcal{T}(u,v), \ u \text{ and } v \in \mathbb{R}\}$, we deduce the invariance relative to $\{\mathcal{T}_u = \mathcal{T}(u,0), \ u \in \mathbb{R}\}$ and the invariance relative to $\{\mathcal{T}_v = \mathcal{T}(0,v), \ v \in \mathbb{R}\}$ and consequently the first integrals $p_1 = c_1$ and $h = h_0$.

Let us impose the *rheonomic constraint* $q_1 = \omega t$. The transformation

$$\mathcal{T}_v: \ [q_1, q_2, \cdots, q_n, t] \longrightarrow [q_1 + \omega v, \ q_2, \cdots, q_n, t + v]$$

makes invariant the Hamilton action. Hence, $\omega p_1 + h = c_1$, where c_1 is a constant associated with $\boldsymbol{Y}^\star \boldsymbol{W} = c_1$, which comes from the Noether theorem applied for the generator vector $\boldsymbol{W}$ such that $\boldsymbol{W}^\star = [\omega, 0, \cdots, 0, 1]$. Let us note that $p_1 = \dfrac{\partial T}{\partial \dot{q}_1} = T_1 + 2\,T_0\,\dot{q}_1 = T_1 + 2\,T_0\,\omega$ for $\dot{q}_1 = \omega$. Scalars $\omega\,p_1$ and h are associated with the first integrals. Therefore,

$$\omega\,(T_1 + 2\,T_0\,\omega) - (T + W) = h_0 \quad \Longrightarrow \quad T_2 - T_0 + W = -h_0,$$

with h_0 constant, which is the expression of the Painlevé theorem.

Application: spherical pendulum. Consider a point P with mass m on a sphere (S) of center O with unit radius and axis denoted by Oz giving the direction of the ascendant vertical.

The kinetic energy in spherical coordinates (Figure 4.2) is

$$T = \frac{1}{2}\,m\left(\dot{\theta}^2 + \dot{\varphi}^2\,\sin^2\theta\right)$$

and the potential energy is

$$W = k^2 \cos\theta.$$

The parameter φ is a secondary parameter; therefore, $\dfrac{\partial T}{\partial \dot{\varphi}} = \alpha$, i.e. $\dot{\varphi}\sin^2\theta = \alpha$, where α is constant. The time t is a secondary parameter; therefore, $T + W = h_0$, where h_0 is constant,

$$\frac{1}{2}\,m\left(\dot{\theta}^2 + \dot{\varphi}^2\,\sin^2\theta\right) + k^2\cos\theta = h_0.$$

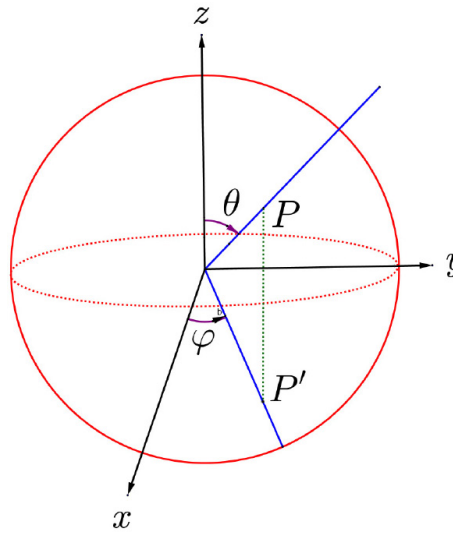


Figure 4.2. *Spherical pendulum: frictionless motion of point P placed on the sphere (S) in the gravitational field*

Now, let us consider the case where we impose the rheonomic constraint $\varphi = \omega\, t$ corresponding to an imposed rotation around axis Oz. Thus,

$$T = \frac{1}{2}\, m \left(\dot{\theta}^2 + \omega^2\, \sin^2\theta\right) \quad \text{and} \quad W = k^2 \cos\theta.$$

Therefore, $(T_2 - T_0 + W) = h_1$, where h_1 is constant. Then,

$$\frac{1}{2}\, m \left(\dot{\theta}^2 - \omega^2\, \sin^2\theta\right) + k^2 \cos\theta = h_0.$$

The equation of energy in the first case differs from Painlevé's equation in the second case.

Now, let us consider a system with Lagrangian

$$G_1 = \frac{1}{2}\, m \left(\frac{\alpha^2}{\sin^2\theta} + \dot{\theta}^2\right) - k^2 \cos\theta \quad \text{with} \quad \dot{\varphi} \sin^2\theta = \alpha.$$

We note that the condition $\dot{\varphi}\sin^2\theta = \alpha$ corresponds to the first integral obtained in the first case, but the study is completely different. Time t is a secondary parameter, and we get the Painlevé's first integral

$$\frac{1}{2}\,m\left(\dot{\theta}^2 + \frac{\alpha^2}{\sin^2\theta}\right) - k^2\cos\theta = h_2, \quad \text{where } h_2 \text{ is constant.} \qquad [4.22]$$

This expression differs from the energy theorem obtained when the parameters θ and φ are independent

$$\frac{1}{2}\,m\left(\dot{\theta}^2 + \frac{\alpha^2}{\sin^2\theta}\right) + k^2\cos\theta = h_0, \quad \text{where } h_0 \text{ is consant.} \qquad [4.23]$$

Indeed, equation [4.22] is associated with the study of the extremals of integral

$$a_1 = \int_{t_0}^{t_1} G_1(\theta, \dot{\theta}, \dot{\varphi})\, dt,$$

submitted to the constraint $\dot{\varphi} = \dfrac{\alpha}{\sin^2\theta}$ or

$$\varphi(t_1) - \varphi(t_0) = \int_{t_0}^{t_1} \frac{\alpha}{\sin^2\theta}\, dt.$$

We have written the expression of the first integral in the Lagrangian expression. For example, when H is independent of q_1, we obtained the first integral $p_1 = c_1$, which is written, by solving relatively to $\dot{q}_1$, in the form

$$\dot{q}_1 = f\left(c_1, q_2, \cdots, q_n, \dot{q}_2, \cdots, \dot{q}_n, t\right). \qquad [4.24]$$

By bringing the result in $G\left(c_1, q_2, \cdots, q_n, \dot{q}_2, \cdots, \dot{q}_n, t\right)$, we obtain

$$G_1\left(q_2, \cdots, q_n, \dot{q}_2, \cdots, \dot{q}_n, t\right) =$$
$$G\left(q_2, \cdots, q_n, f\left(c_1, q_2, \cdots, q_n, \dot{q}_2, \cdots, \dot{q}_n, t\right), \dot{q}_2, \cdots, \dot{q}_n, t\right).$$

Even if we take into account the partial result $\dot{q}_1 = f\left(c_1, q_2, \cdots, q_n, \dot{q}_2, \cdots, \dot{q}_n, t\right)$, the study associated with system $(\mathcal{S}_1)$ corresponding to G_1 is not equivalent to the study associated with the system $(\mathcal{S})$ corresponding to G. This observation leads to the problem of the re-injection of the partial result into the Lagrangian.

4.7. Re-injection of a partial result

What modification of the Lagrangian must be made when we know the expression of a first linear integral of the motion? Let a material system with n independent parameters $q_1, \cdots, q_n$, with q_1 being a secondary parameter. The Lagrangian expression is $G = G(q_2, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)$. The expression of the first integral corresponding to q_1 is $\dfrac{\partial T}{\partial \dot{q}_1} = \alpha$. Let

$$\boldsymbol{Q}_0 = \begin{bmatrix} q_1(t_0) \\ \vdots \\ q_n(t_0) \end{bmatrix} \quad \text{and} \quad \boldsymbol{Q}_1 = \begin{bmatrix} q_1(t_1) \\ \vdots \\ q_n(t_1) \end{bmatrix}$$

be the given ends of the motion of the system $(\mathcal{S})$. The equations of motion are associated with the extremal curves connecting $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ in integral

$$a = \int_{t_0}^{t_1} G(q_2, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t)\, dt.$$

They are the same curves that make extremum integral

$$a = \int_{t_0}^{t_1} \left[G(q_2, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t) - \alpha\, \dot{q}_1 \right] dt,$$

where α is constant. Indeed, let us write $a_1 = a - \alpha\, q_1(t_1) + \alpha\, q_1(t_0)$, scalars a_1 and a differ from a constant. Let us choose the value of α given by the first integral

$$\frac{\partial T}{\partial \dot{q}_1} = \alpha.$$

Then,

$$\delta a_1 = \int_{t_0}^{t_1} \left[\left(\frac{\partial G}{\partial \dot{q}_1} - \alpha \right) \delta \dot{q}_1 + \sum_{j=2}^{n} \frac{\partial G}{\partial \dot{q}_j}\, \delta \dot{q}_j + \frac{\partial G}{\partial q_j}\, \delta q_j \right] dt$$

$$= \sum_{j=2}^{n} \int_{t_0}^{t_1} \left[-\frac{d}{dt} \left(\frac{\partial G}{\partial \dot{q}_j} \right) + \frac{\partial G}{\partial q_j} \right] \delta q_1\, dt + \left[\sum_{j=2}^{n} \frac{\partial G}{\partial \dot{q}_j}\, \delta q_1 \right]_{\tilde{\boldsymbol{Q}}(t_0)}^{\tilde{\boldsymbol{Q}}(t_1)}$$

where $\tilde{\boldsymbol{Q}} = \begin{bmatrix} q_2 \\ \vdots \\ q_n \end{bmatrix}$ is fixed at t_0 and t_1, and $\boldsymbol{Q} = \begin{bmatrix} q_1 \\ \tilde{\boldsymbol{Q}} \end{bmatrix}$. Thus, $\left[\sum_{j=2}^{n} \frac{\partial G}{\partial \dot{q}_j} \delta q_1 \right]_{\tilde{\boldsymbol{Q}}(t_0)}^{\tilde{\boldsymbol{Q}}(t_1)}$ is zero and

$$\delta a_1 = \int_{t_0}^{t_1} \sum_{j=2}^{n} \left[-\frac{d}{dt} \left(\frac{\partial G}{\partial \dot{q}_j} \right) + \frac{\partial G}{\partial q_j} \right] \delta q_1 \, dt.$$

Therefore, the equations of motion are

$$\frac{d}{dt} \left(\frac{\partial G}{\partial \dot{q}_j} \right) + \frac{\partial G}{\partial q_j} = 0 \text{ for } j \in \{2, \cdots, n\},$$

to which we add the condition $\dfrac{\partial G}{\partial \dot{q}_1} = \alpha$. They are the equations of motion of $(\mathcal{S})$.

CONCLUSION 4.2.– If $G_1 = G(q_2, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t) - \alpha \ \dot{q}_1$ with $\dfrac{\partial G}{\partial \dot{q}_1} = \alpha$, Hamilton's principle is equivalent to a Hamilton's principle for the $n-1$ parameters $q_2, \cdots, q_n$, where $q_2, \cdots, q_n$ are fixed for t_0 and t_1.

EXAMPLE 4.5.– We consider the spherical pendulum described in section 4.6.2. The parameter φ is a secondary one, and $p_\varphi = \dot{\varphi} \sin^2 \theta = \alpha$.

We consider $G_1 = G - \alpha \dot{\varphi} = \dfrac{1}{2} m \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) - k^2 \cos \theta - \alpha \dot{\varphi}$. By re-injecting the first integral p_φ in G_1, we obtain a Lagrangian only depending on parameter θ,

$$G_1 = \frac{1}{2} m \left(\dot{\theta}^2 - \frac{\alpha^2}{\sin^2 \theta} \right) - k^2 \cos \theta,$$

corresponding to a system with kinetic and potential energies as

$$T = \frac{1}{2} m \dot{\theta}^2 \quad \text{and} \quad W = k^2 \cos \theta + \frac{m \alpha^2}{2 \sin^2 \theta}.$$

Then, we obtain the energy first integral

$$\frac{1}{2} m \left(\dot{\theta}^2 + \frac{\alpha^2}{\sin^2 \theta} \right) + k^2 \cos \theta = h_0,$$

or, by taking p_φ into account,

$$\frac{1}{2} m \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) + k^2 \cos \theta = h_0,$$

which is the expression obtained in [4.23].

4.8. The Maupertuis principle

Let us use the results from section 4.7 when t is a secondary parameter. Consider system $(\mathcal{S})$, whose Lagrangian $G = G(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ is explicitly independent of time t. The homogeneous Lagrangian is

$$L(\boldsymbol{X}, \boldsymbol{X}') = G(\boldsymbol{Q}, \dot{\boldsymbol{Q}})\, t'$$

where $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix}$ is a function of parameter τ. The system is associated with Painlevé's first integral, which has the dimension of an energy,

$$h = \frac{\partial L}{\partial t'} = -E \quad \text{with} \quad \frac{\partial L}{\partial t'} = G - \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\dot{\boldsymbol{Q}}$$

or

$$H(\boldsymbol{Q}, \boldsymbol{P}) = \boldsymbol{E}.$$

Since $\dfrac{\partial L}{\partial t'} = -E$ corresponds to the linear first integral associated with the secondary parameter t, according to section 4.7 searching for the extremals of the Hamilton action is equivalent to the research of the extremals of

$$a = \int_{\boldsymbol{X}_0}^{\boldsymbol{X}_1} L(\boldsymbol{X}, \boldsymbol{X}')\, d\tau, \quad \text{where} \quad \boldsymbol{X}_0 = \begin{bmatrix} \boldsymbol{Q}(t_0) \\ t_0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{X}_1 = \begin{bmatrix} \boldsymbol{Q}(t_1) \\ t_1 \end{bmatrix}$$

and they are also the extremals of

$$a_1 = \int_{\boldsymbol{X}_0}^{\boldsymbol{X}_1} \left[L(\boldsymbol{X}, \boldsymbol{X}') - (-E)t' \right] d\tau \quad \text{with} \quad \frac{\partial L}{\partial t'} = -E,$$

or equivalently,

$$a_1 = \int_{t_0}^{t_1} \left[G(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) + E \right] dt \quad \text{with} \quad \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\dot{\boldsymbol{Q}} - G = E.$$

Therefore, since $\boldsymbol{P}^\star = \dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}$,

$$G + E = \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\dot{\boldsymbol{Q}} \quad \Longrightarrow \quad (G + E)\, dt = \boldsymbol{P}^\star \boldsymbol{dQ}.$$

In an n-dimensional space, we are brought back to the extremals of

$$a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} \boldsymbol{P}^\star d\boldsymbol{Q} \quad \text{with} \quad H(\boldsymbol{Q}, \boldsymbol{P}) = E.$$

THEOREM 4.3.– ***The Maupertuis principle***. The trajectories of a system whose Lagrangian (or Hamiltonian) is explicitly independent of time are obtained by the extremal curves' research of integral

$$a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} \boldsymbol{P}^\star d\boldsymbol{Q},$$

where $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ are the ends of the curves, with $H(\boldsymbol{Q}, \boldsymbol{P}) = E$, where E is the constant energy along the trajectories.

4.8.1. *First application: case of a material point*

A material point, with mass m in the three-dimensional Euclidean space $\mathcal{E}^3$, admits a kinetic energy T and a potential energy W. The time is a secondary parameter. We have the energy first integral $T + W = E$. Since $G + E = 2\,T$, we obtain

$$a_1 = \int_{t_0}^{t_1} 2\,T\,dt$$

and the Maupertuis principle implies that the trajectories make extremal the time integral of kinetic energy.

From $2\,T = 2(E - W)$ and $T = \dfrac{1}{2}\,m\left(\dfrac{ds}{dt}\right)^2$, with s denoting the positively oriented curvilinear abscissa of the trajectory, we obtain

$$\frac{1}{2}\,m\left(\frac{ds}{dt}\right)^2 = E - W \quad \Longrightarrow \quad \frac{ds}{dt} = \sqrt{\frac{2\,(E-W)}{m}}.$$

Hence,

$$a_1 = \int_{s_0}^{s_1} \sqrt{2m\,(E-W)}\,ds \quad \text{where} \quad s_0 = s(t_0) \quad \text{and} \quad s_1 = s(t_1).$$

If a refractive index n at any point M of $\mathcal{E}^3$ is defined by using the relation

$$n(M) = \sqrt{2m\,(E-W)}, \quad \text{we are back to integral} \quad a_1 = \int_{s_0}^{s_1} n(M)ds.$$

COROLLARY 4.1.– The trajectories of point M, with mass m and potential energy $W(M)$, are the optical paths associated with the refractive index $n(M) = \sqrt{2m\,(E - W)}$, where E is the constant total energy of point M.

It can be noted that $W(M) = E - \dfrac{n^2(M)}{2\,m}$.

COROLLARY 4.2.– To a given optical medium with refractive index $n(M)$ and to optical paths in the medium, we can associate a material point M of mass m, total energy E and potential energy $W(M) = E - \dfrac{n^2(M)}{2\,m}$ such that the optical paths correspond to the trajectories of point M.

The corollary is the starting point for Louis de Broglie's theory connecting wave mechanics and geometric optics (De Broglie 1924). It can be noted that the writing of the Maupertuis principle corresponds to finding the extremals of $a_1 = \displaystyle\int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} \boldsymbol{P}^\star d\boldsymbol{Q}$ with $\dfrac{1}{2}\dfrac{\boldsymbol{P}^\star \boldsymbol{P}}{m} + W = E$ and $\boldsymbol{Q}$ being the column of the Cartesian coordinates of M. Thus, $\boldsymbol{P}^\star \boldsymbol{P} = 2\,m\,(E - W)$ or $\boldsymbol{P}^\star \boldsymbol{P} = n^2(M)$, scalar field $n(M)$ being the refractive index. The extremal curves satisfy the differential condition written in Part 1, Chapter 2 in the form

$$\frac{dn\boldsymbol{T}}{ds} = \text{gradn}.$$

4.8.2. *Second application: introduction to the Riemannian geometry*

Let $(\mathcal{S})$ be a system with n independent parameters and homogeneous second-degree kinetic energy T with respect to $\dot{q}_1, \cdots, \dot{q}_n$. We can write $2\,T = \dot{\boldsymbol{Q}}^\star \mathcal{A}\, \dot{\boldsymbol{Q}}$, where $\mathcal{A}$ is a symmetric matrix only depending on $\boldsymbol{Q}$ and t. Then,

$$2\,T = \sum_{ij} a_{ij}\, \dot{q}_i \dot{q}_j \quad \text{with} \quad a_{ij} = a_{ji}\,,\ i, j \in \{1, \cdots, n\}\,.$$

The forces applied to $(\mathcal{S})$ derive from the potential energy

$$W = W(\boldsymbol{Q}, t) = W(q_1, \cdots, q_n, t).$$

If the Lagrangian is explicitly independent of t, matrix $\mathcal{A}$ only depends on $\boldsymbol{Q}$ and $W = W(q_1, \cdots, q_n)$. Energy $T + W = E$ is a first integral of the motion. In the space of parameters $(q_1, \cdots, q_n)$, the trajectories of system $(\mathcal{S})$ are the extremals of

$$a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} (G + E)\, dt = \int_{t_0}^{t_1} 2\,T\, dt = \int_{t_0}^{t_1} 2\,(E - W)\, dt.$$

For an appropriate orientation of the trajectories in the space of parameters,

$$\dot{\boldsymbol{Q}}^{\star}\,\mathcal{A}\,\dot{\boldsymbol{Q}} = 2\,(E-W) \implies d\boldsymbol{Q}^{\star}\mathcal{A}\,d\boldsymbol{Q}$$

$$= 2\,(E-W)\,(dt)^2 \implies dt = \sqrt{\frac{d\boldsymbol{Q}^{\star}\mathcal{A}\,d\boldsymbol{Q}}{2\,(E-W)}},$$

hence,

$$a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} \sqrt{2\,(E-W)\,(d\boldsymbol{Q}^{\star}\mathcal{A}\,d\boldsymbol{Q})}.$$

The mapping

$$\boldsymbol{Q} \in \mathbb{R}^n \longrightarrow \mathcal{B}(\boldsymbol{Q}) = 2\,(E-W(\boldsymbol{Q}))\,\mathcal{A}(\boldsymbol{Q})$$

represents a field of symmetric matrices whose coefficients are defined at any point $\boldsymbol{Q}$ by $b_{ij} = 2\,(E-W)\;a_{ij},\; i,j \in \{1,\cdots,n\}$. The matrix field is associated with a field of positive definite quadratic forms.

For example, for $2\,(E-W(\boldsymbol{Q}))\,\mathcal{A}(\boldsymbol{Q}) = \mathbf{1}$, we associate $a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} \sqrt{d\boldsymbol{Q}^{\star}d\boldsymbol{Q}}$, which represents the length of the trajectory in the space of parameters. The space of parameters becomes an n-dimensional Euclidean vectorial space.

DEFINITION 4.9.– ***A Riemann space*** is an n-dimensional vectorial space provided with a field of positive definite quadratic forms.

At any point, the matrix $\mathcal{B}$ represents the quadratic form in the vectorial space basis. If we consider a vector field

$$\boldsymbol{Q} \in \mathbb{R}^n \longrightarrow \boldsymbol{V}(\boldsymbol{Q}),$$

we call the scalar product in $\boldsymbol{Q}$ of the vector field $\boldsymbol{V}$

$$\boldsymbol{V}^{\star}(\boldsymbol{Q})\,\mathcal{B}(\boldsymbol{Q})\,\boldsymbol{V}(\boldsymbol{Q}).$$

If in the previous study we denote $d\sigma = \sqrt{2\,(E-W)\,d\boldsymbol{Q}^{\star}\mathcal{A}\,d\boldsymbol{Q}}$, we obtain

$$a_1 = \int_{\boldsymbol{Q}_0}^{\boldsymbol{Q}_1} d\sigma.$$

We say that the trajectory curves of system (S) are the length extremals for a Riemann space whose metric is associated with both kinetic energy and potential energy.

5

Jacobi's Integration Method

5.1. Canonical transformations

The Hamilton equations

$$\begin{cases} \dfrac{d\boldsymbol{Q}}{dt} = \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} \\ \dfrac{d\boldsymbol{P}}{dt} = -\left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} \end{cases} \qquad [5.1]$$

can be obtained by writing that

$$a = \int_{(\boldsymbol{Q}_0, t_0)}^{(\boldsymbol{Q}_1, t_1)} \boldsymbol{P}^{\star} d\boldsymbol{Q} + h\, dt$$

is extremum, where $(\boldsymbol{Q}_0, t_0)$ and $(\boldsymbol{Q}_1, t_1)$ are given and $h, \boldsymbol{P}, \boldsymbol{Q}$ and t are related through the relation $h + H(\boldsymbol{P}, \boldsymbol{Q}, t) = 0$. Equations [5.1] are called *canonical form of motion equations*.

We search for a change of variables in $\mathbb{R}^{2n} : (\boldsymbol{P}, \boldsymbol{Q}, t) \longrightarrow (\boldsymbol{A}, \boldsymbol{B}, t)$ represented by

$$\begin{cases} \boldsymbol{A} = \mathcal{A}(\boldsymbol{P}, \boldsymbol{Q}, t) \\ \boldsymbol{B} = \mathcal{B}(\boldsymbol{P}, \boldsymbol{Q}, t) \end{cases} \text{ or equivalently } \begin{cases} \boldsymbol{P} = \mathcal{P}(\boldsymbol{A}, \boldsymbol{B}, t) \\ \boldsymbol{Q} = \mathcal{Q}(\boldsymbol{A}, \boldsymbol{B}, t) \end{cases} \qquad [5.2]$$

which can be made explicit in the form

$$\begin{cases} a_i = \mathcal{A}_i(p_1, \cdots, p_n, q_1, \cdots, q_n, t) \\ b_i \ = \mathcal{B}_i(p_1, \cdots, p_n, q_1, \cdots, q_n, t) \end{cases} \quad \text{with} \quad i \in \{1, \cdots, n\}\,.$$

Generally, the change of variables does not retain the canonical form of equations [5.1].

DEFINITION 5.1.– The change of variables – called the Jacobi transformation – is said to be canonical if and only if equations [5.1] are transformed as

$$\begin{cases} \dfrac{d\boldsymbol{B}}{dt} = \left(\dfrac{\partial K}{\partial \boldsymbol{A}}\right)^{\star} \\ \dfrac{d\boldsymbol{A}}{dt} = -\left(\dfrac{\partial K}{\partial \boldsymbol{B}}\right)^{\star} \end{cases} \qquad [5.3]$$

where $K(\boldsymbol{A}, \boldsymbol{B}, t)$ replaces the Hamiltonian $H(\boldsymbol{P}, \boldsymbol{Q}, t)$ and represents the new Hamiltonian as a function of $(\boldsymbol{A}, \boldsymbol{B}, t)$.

Equations [5.3] express that

$$b = \int_{(\boldsymbol{B}_0, t_0)}^{(\boldsymbol{B}_1, t_1)} \boldsymbol{A}^{\star} d\boldsymbol{B} - K(\boldsymbol{A}, \boldsymbol{B}, t)\, dt$$

is extremum on a curve (C) connecting points $\boldsymbol{B}_0 = \boldsymbol{B}(t_0)$ and $\boldsymbol{B}_1 = \boldsymbol{B}(t_1)$.

The change of variables is canonical if a and b are simultaneously extremal. This is the case for the $a - b$ constant. So, a sufficient condition is that the differential form

$$\boldsymbol{P}^{\star} d\boldsymbol{Q} - H dt - (\boldsymbol{A}^{\star} d\boldsymbol{B} - K dt)$$

is exact, i.e. if there exists a function S such that

$$dS = \boldsymbol{P}^{\star} d\boldsymbol{Q} - H dt - (\boldsymbol{A}^{\star} d\boldsymbol{B} - K dt)\,. \qquad [5.4]$$

If $\det\left\{\dfrac{\partial^2 S}{\partial q_i \partial q_j}\right\} \neq 0$, *function S is called the* ***generating function of the canonical transformation***.

Given relation [5.4], the generating function S satisfies the conditions

$$\begin{cases} \boldsymbol{P}^\star = \dfrac{\partial S(\boldsymbol{Q},\boldsymbol{B},t)}{\partial \boldsymbol{Q}} \\ -\boldsymbol{A}^\star = \dfrac{\partial S(\boldsymbol{Q},\boldsymbol{B},t)}{\partial \boldsymbol{B}} \\ K - H = \dfrac{\partial S(\boldsymbol{Q},\boldsymbol{B},t)}{\partial t} \end{cases} \quad \text{or} \quad \begin{cases} p_i = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial q_i} \\ -a_i = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial b_i} \\ K(a_1,\cdots,a_n,b_1,\cdots,b_n,t) \\ -H(p_1,\cdots,p_n,q_1,\cdots,q_n,t) \\ = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial t} \end{cases} \quad [5.5]$$

with $i \in \{1,\cdots,n\}$. The equations giving $p_1,\cdots,p_n$ in system [5.5] constitute an implicit system of n equations, where $b_1,\cdots,b_n$ are function of $p_1,\cdots,p_n$, $q_1,\cdots,q_n$ and t, or

$$\begin{cases} p_1 = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial q_1}, \\ \vdots \\ p_i = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial q_i}, \\ \vdots \\ p_n = \dfrac{\partial S(q_1,\cdots,q_n,b_1,\cdots,b_n,t)}{\partial q_n}. \end{cases}$$

Since $\det\left\{\dfrac{\partial^2 S}{\partial q_i \partial q_j}\right\} \neq 0$, the theorem of implicit functions makes it possible to deduce $\boldsymbol{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ in the form $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{P},\boldsymbol{Q},\boldsymbol{t})$. Then, $\boldsymbol{B}$ being obtained, we can obtain

$$\boldsymbol{A}^\star = -\frac{\partial S(\boldsymbol{Q},\boldsymbol{B}(\boldsymbol{P},\boldsymbol{Q},\boldsymbol{t}),t)}{\partial \boldsymbol{B}},$$

where $\boldsymbol{A} = \mathcal{A}(\boldsymbol{P},\boldsymbol{Q},t)$ corresponds to the change of variables [5.2], which becomes canonical.

In new variables $\boldsymbol{A}$ and $\boldsymbol{B}$, the expression of S implies

$$K(\boldsymbol{A},\boldsymbol{B},t) = \frac{\partial S\left(\boldsymbol{Q}(\boldsymbol{A},\boldsymbol{B},t),\boldsymbol{B},t\right)}{\partial t} + H\left(\boldsymbol{P}(\boldsymbol{A},\boldsymbol{B},t),\boldsymbol{Q}(\boldsymbol{A},\boldsymbol{B},t),t\right) \quad [5.6]$$

and it is possible to write the system of the Hamilton equations [5.3] with the new variables.

5.2. The Jacobi method

We search for generating functions such that $K \equiv 0$.

The equations of motion in the new variables are

$$\frac{d\boldsymbol{A}}{dt} = 0 \quad \text{and} \quad \frac{d\boldsymbol{B}}{dt} = 0,$$

or

$$\boldsymbol{A} = \boldsymbol{A}_0 \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{B}_0, \tag{5.7}$$

where $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ are constant vectors, i.e. $a_1, \cdots, a_n, b_1, \cdots, b_n$ are constants depending on the initial conditions.

Given relation [5.6], the equation giving the generating function $S = S(\boldsymbol{Q}, \boldsymbol{B}, t)$ can be written as

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial \boldsymbol{Q}}, \boldsymbol{Q}, t\right) = 0 \tag{5.8}$$

where H is the Hamiltonian. This first-order partial differential equation is called the *Jacobi equation*. It is written

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S\left(q_1, \cdots, q_n, b_1, \cdots, b_n, t\right)}{\partial q_1}, \right.$$
$$\left. \cdots, \frac{\partial S\left(q_1, \cdots, q_n, b_1, \cdots, b_n, t\right)}{\partial q_n}, q_1, \cdots, q_n, t\right) = 0.$$

The equations of motion are in the form [5.7] for any solution $S = S\left(q_1, \cdots, q_n, \ b_1, \cdots, b_n, t\right)$ of equation [5.8] depending on n arbitrary constants denoted by $b_1, \cdots, b_n$ and such that $\det\left\{\dfrac{\partial^2 S}{\partial q_i \partial q_j}\right\} \neq 0$; the change of canonical variables is associated with the system

$$\begin{cases} \boldsymbol{P}^\star = \dfrac{\partial S\left(\boldsymbol{Q}, \boldsymbol{B}, t\right)}{\partial \boldsymbol{Q}} \\ -\boldsymbol{A}^\star = \dfrac{\partial S\left(\boldsymbol{Q}, \boldsymbol{B}, t\right)}{\partial \boldsymbol{B}} \end{cases}$$

which expresses the equations of motion in variables $\boldsymbol{P}$ and $\boldsymbol{Q}$ as a function of the $2n$ integration constants $[a_1, \cdots, a_n]^\star = \boldsymbol{A}$ and $[b_1, \cdots, b_n]^\star = \boldsymbol{B}$ by solving the system in the form

$$\boldsymbol{Q} = \mathcal{Q}(\boldsymbol{A}, \boldsymbol{B}, t) \quad \text{and} \quad \boldsymbol{P} = \mathcal{P}(\boldsymbol{A}, \boldsymbol{B}, t).$$

$\boldsymbol{Q} = \mathcal{Q}(\boldsymbol{A}, \boldsymbol{B}, t)$ represents the equations of motion in the form $q_i = q_i(a_1, \cdots, a_n, b_1, \cdots, b_n, t)$,

$\boldsymbol{P} = \mathcal{P}(\boldsymbol{A}, \boldsymbol{B}, t)$ represents the impulsion covector obtained after replacing $\boldsymbol{Q}$ as a function of $\boldsymbol{A}, \boldsymbol{B}$ and t.

REMARK 5.1.– If S is a solution of [5.8], $S + \lambda$, where $\lambda \in R$, is also a solution of [5.8]. Nevertheless, λ cannot be taken as one of the parameters $b_1, \cdots, b_n$ since $\dfrac{\partial(S+\lambda)}{\partial\lambda} = 1$ and $\forall i \in \{1, \cdots, n\}$, $\dfrac{\partial^2(S+\lambda)}{\partial\lambda\partial qi} = 0$. In this case, $\det\left\{\dfrac{\partial^2(S+\lambda)}{\partial q_i \partial q_j}\right\} = 0$, where λ is one of the parameters q_i.

EXAMPLE 5.1.– ***One-dimensional harmonic oscillator***. Consider a point M with mass unity, on the axis $O\boldsymbol{i}$, attracted by the point O proportionally to the distance. We denote this $q = \overline{OM}_{\boldsymbol{i}}$. The kinetic energy and the potential energy are

$$T = \frac{1}{2}\,\dot{q}^2 \quad \text{and} \quad W = \frac{1}{2}\,\omega^2 q^2.$$

Consequently,

$$a = \frac{1}{2}\int_{t_0}^{t_1} \left(\dot{q}^2 - \omega^2 q^2\right)\, dt \quad \text{hence} \quad \begin{cases} p = \dot{q} \\ \\ h = -\dfrac{1}{2}\,\dot{q}^2 - \dfrac{1}{2}\,\omega^2 q^2 = -H(p, q, t) \end{cases}.$$

We also deduce this relation from $H = T + W$ and $H(p, q, t) = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$. The Jacobi equation becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}\,\omega^2 q^2 = 0. \qquad [5.9]$$

We search for the solutions of [5.9] in the form $S = \beta\, t + S_1(q)$, where β is a real constant. Thus,

$$\beta + \frac{1}{2}\,S_1'^2(q) + \frac{1}{2}\,\omega^2 q^2 = 0,$$

and a solution is

$$S_1 = \int_{q_0}^{q} \sqrt{-2\beta - \omega^2 q^2}\, dq,$$

which yields

$$S = \beta t + \int_{q_0}^{q} \sqrt{-2\beta - \omega^2 q^2}\, dq.$$

We verify that we have obtained a solution depending on one parameter β. Then,

$$\frac{\partial S}{\partial \beta} = -\alpha,$$

where α is constant, expresses the equation of motion in the form

$$t + \alpha - \int_{q_0}^{q} \frac{dq}{\sqrt{-2\beta - \omega^2 q^2}} = 0.$$

Let us denote $a^2\omega^2 = -2\beta$, then

$$\omega\,(t + \alpha) = \int_{q_0}^{q} \frac{dq}{\sqrt{a^2 - q^2}}.$$

By writing $\alpha = -t_0$, we obtain $q = a\cos\omega\,(t - t_0)$.

5.2.1. *One position parameter is secondary*

The parameter q_1 is assumed to be secondary.

Then, $\dfrac{\partial H}{\partial q_1} = 0$. We search for S in the form

$$S = \beta_1 q_1 + S_1\,(q_{2,\cdots}, q_n, t, \beta_1)\,.$$

We deduce

$$H\,(p_1, \cdots, p_n, q_2, \cdots, q_n, t) = H\left(\beta_1, \frac{\partial S_1}{\partial q_2}, \cdots, \frac{\partial S_1}{\partial q_n}, q_2, \cdots, q_n, t\right),$$

where β_1 is constant and S_1 verifies

$$\frac{\partial S_1}{\partial t} + H\left(\beta_1, \frac{\partial S_1}{\partial q_2}, \cdots, \frac{\partial S_1}{\partial q_n}, q_2, \cdots, q_n, t\right) = 0,$$

which is the new form of Jacobi's equation.

5.2.2. *Time is a secondary parameter*

This is the case when $\dfrac{\partial H}{\partial t} = 0$ or $H = H(\boldsymbol{P}, \boldsymbol{Q})$. We search for the generating function S in the form

$$S = -Et + S_1\left(q_{1,\cdots}, q_n, E\right).$$

The Jacobi equation becomes

$$H\left(\frac{\partial S_1}{\partial q_1}, \cdots, \frac{\partial S_1}{\partial q_n}, q_1, \cdots, q_n\right) = E$$

where E is the Painlevé energy of the system.

Let $S_1 = S_1\left(q_{1,\cdots}, q_n, E, \beta_2, \cdots, \beta n\right)$ be a solution depending on the $n-1$ parameters $\beta_2, \cdots, \beta n$. The equations of motion become

$$\frac{\partial S}{\partial E} = -t_0 \text{ or } t - t_0 = \frac{\partial S_1}{\partial E}$$
$$\frac{\partial S}{\partial \beta i} + \alpha_i = 0 \quad \text{with} \quad i \in \{2, \cdots, n\},$$

where t_0 and α_i are constant. The first relation expresses the law of time on the trajectory. The $n-1$ other equations are independent of time and connect the parameters $q_1, \cdots, q_n$; they represent the equations of the trajectory.

5.3. The material point in various systems of representation

In the three-dimensional space, consider the motion of a material point with mass m, subject to a force field independent of time and deriving from a potential W. We consider a form of W when we can integrate the Jacobi equation with separated position parameters.

5.3.1. *Case of Cartesian coordinates*

The kinetic energy and the Hamiltonian are

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \quad \text{and} \quad H = \frac{1}{2m}\left[p^2 + q^2 + r^2\right] + W(x, y, z),$$

where $p = m\,\dot{x}$, $q = m\,\dot{y}$, $r = m\,\dot{z}$ denote the conjugate variables of x, y, z, respectively. Taking into account that t is a secondary parameter, the generating function is $S = -E\,t + F(x, y, z)$ and the Jacobi equation can be written as

$$\frac{1}{2m}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right] + W(x, y, z) = E.$$

In order for variables x, y, z to be separated, we must have

$$W(x, y, z) = W_1(x) + W_2(y) + W_3(z).$$

By writing $S = -E\,t + F_1(x) + F_2(y) + F_3(z)$, we obtain

$$\left[\left(\frac{dF_1}{dx}\right)^2 + 2m\,W_1(x)\right] + \left[\left(\frac{dF_2}{dy}\right)^2 + 2m\,W_2(y)\right] + \left[\left(\frac{dF_3}{dz}\right)^2 + 2m\,W_3(z)\right] = 2mE.$$

Since each of the three terms within the square brackets depends on only one variable, we can choose each term constant,

$$\left(\frac{dF_1}{dx}\right)^2 + 2m\,W_1(x) = \alpha, \quad \left(\frac{dF_2}{dy}\right)^2 + 2m\,W_2(y) = \beta,$$
$$\left(\frac{dF_3}{dz}\right)^2 + 2m\,W_3(z) = \gamma,$$

with $\alpha + \beta + \gamma = 2m\,E$. Each of the three expressions is integrable and

$$S = -E\,t + \int_x \sqrt{\alpha - 2m\,W_1(x)}\,dx + \int_y \sqrt{\beta - 2m\,W_2(y)}\,dy + \int_z \sqrt{\gamma - 2m\,W_3(z)}\,dz.$$

5.3.2. *Case of cylindrical representation*

The kinetic energy and the Hamiltonian are

$$T = \frac{1}{2}\,m\left(\dot{\rho}^2 + \rho^2\,\dot{\theta}^2 + \dot{z}^2\right) \quad \text{and} \quad H = \frac{1}{2m}\left[p^2 + \frac{q^2}{\rho^2} + r^2\right] + W(\rho, \theta, z),$$

where $p = m\,\dot{\rho}$, $q = m\,\rho^2\dot{\theta}$, $r = m\,\dot{z}$ denote the conjugate variables of ρ, θ, z, respectively. Taking into account that t is a secondary parameter, the generating function is

$$S = -E\,t + F(\rho, \theta, z)$$

and the Jacobi equation writes

$$\frac{1}{2m}\left[\left(\frac{\partial F}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial F}{\partial \theta}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right] + W(\rho, \theta, z) = E.$$

If we choose W in the form

$$W(\rho, \theta, z) = W_1(z) + W_2(\rho, \theta),$$

we write $S = -E\,t + F_1(z) + G(\rho, \theta)$ and the Jacobi equation can be separated into two equations,

$$\left(\frac{dF_1}{dz}\right)^2 + 2m\,W_1(z) = 2m\,\alpha$$

and

$$\left(\frac{\partial G}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial G}{\partial \theta}\right)^2 + 2m\,W_2(\rho, \theta) = 2m\,(E - \alpha),$$

where α is constant. The first equation can be integrated. Multiplying the second equation by ρ^2, we obtain

$$\rho^2\left(\frac{\partial G}{\partial \rho}\right)^2 + \left(\frac{\partial G}{\partial \theta}\right)^2 + 2m\,\rho^2\,W_2(\rho, \theta) - 2m\,(E - \alpha)\rho^2 = 0.$$

If we write $W_2(\rho, \theta) = h(\rho) + \dfrac{g(\theta)}{\rho^2}$, the second equation can be separated into

$$\rho^2\left(\frac{\partial G}{\partial \rho}\right)^2 + 2m\,\rho^2\,h(\rho) - 2m\,(E - \alpha)\rho^2 = 2m\,\beta$$

and

$$\left(\frac{\partial G}{\partial \theta}\right)^2 + 2m\,g(\theta) = -2m\,\beta,$$

where β is constant. Hence, $W(\rho,\theta,z) = W_1(z) + h(\rho) + \dfrac{1}{\rho^2}\, g(\theta)$, and by separating the three variables z, θ, ρ, we obtain the generating Jacobi function

$$S = -E\,t + \sqrt{2m}\left(\int_z \sqrt{\alpha - W_1(z)}\,dz + \int_\theta \sqrt{-(\beta + g(\theta))}\,d\theta + \int_\rho \sqrt{E - \alpha + \frac{\beta}{\rho^2} - h(\rho)}\,d\rho\right).$$

5.3.3. *Case of spherical representation*

The kinetic energy and the Hamiltonian are

$$T = \frac{1}{2}\,m\left(\dot r^2 + r^2\,\dot\theta^2 + r^2\,\dot\varphi^2\,\sin^2\,\theta\right)$$

and

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\,\sin^2\,\theta}\right] + W(r,\theta,\varphi),$$

where $p_r = m\,\dot r,\ p_\theta = m\,r^2\,\dot\theta,\ p_\varphi = m\,r^2\dot\varphi\,\sin^2\,\theta$ denote the conjugate variables of r, θ, φ. Taking into account that t is a secondary parameter, the generating function is of the form $S = -E\,t + F(r,\theta,\varphi)$ and the Jacobi equation can be written as

$$\frac{1}{2m}\left[r^2\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{\partial F}{\partial \theta}\right)^2 + \frac{1}{\sin^2\,\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right] + r^2\,W(r,\theta,\varphi) = E\,r^2.$$

If we choose W such that $r^2\,W(r,\theta,\varphi) = W_1(r) + W_2(\theta,\varphi)$, the generating function can be written as

$$S = -E\,t + F_1(r) + G(\theta,\varphi)$$

and the Jacobi equation can be separated into two equations

$$r^2\left(\frac{dF_1}{dr}\right)^2 + 2m\,W_1(r) - 2m\,E\,r^2 = 2m$$

and

$$sin^2\,\theta\left(\frac{\partial G}{\partial \theta}\right)^2 + \left(\frac{\partial G}{\partial \varphi}\right)^2 + 2m\,\sin^2\theta\,W_2(\theta,\varphi) = -2m\,\alpha\,sin^2\,\theta,$$

where α is constant. The first equation can be integrated. If we write $W_2(\theta,\varphi) = g(\theta) + \dfrac{1}{\sin^2\theta}\,h(\varphi)$, then $G(\theta,\varphi) = F_2(\theta) + F_3(\varphi)$ with

$$\left(\frac{dF_2}{d\theta}\right)^2 = 2m\left[\frac{\beta}{sin^2\,\theta} - \alpha - g(\theta)\right] \quad \text{and} \quad \left(\frac{dF_3}{d\varphi}\right)^2 = 2m\left[-\beta - h(\varphi)\right]$$

and we obtain the generating function which separates the three variables r, θ, φ

$$S = -E\,t + \sqrt{2m}\int_r \sqrt{\frac{\alpha}{r^2} - W_1(r) + E}\,dr$$
$$+\sqrt{2m}\int_\theta \sqrt{\frac{\beta}{\sin^2\,\theta} - \alpha - g(\theta)}\,d\theta + \sqrt{2m}\int_\varphi \sqrt{-\beta - h(\varphi)}\,d\varphi.$$

As shown in section 5.2, we obtain the time law and the trajectories in the three cases.

We can also consider more complex cases such as parabolic coordinates or elliptical coordinates[1].

5.4. Case of the Liouville integrability

DEFINITION 5.2.– A system $(\mathcal{S})$, with n independent parameters, has a kinetic energy of the form

$$T = \frac{1}{2}\left(U_1 + \cdots + U_n\right)\left(V_1\,\dot{q}_1^2 + \cdots + V_n\,\dot{q}_n^2\right),$$

1 **Parabolic coordinates** (ζ,η,φ):

Based on the cylindrical representation, we write:

$z = \dfrac{1}{2}(\zeta - \eta)$ and $\rho = \sqrt{\zeta\,\eta}$. We obtain $W(\zeta,\eta,\varphi) = \dfrac{f(\varphi)}{\zeta\,\eta} + \dfrac{g(\zeta)+h(\eta)}{\zeta+\eta}$.
The surfaces $\zeta = Cte$ and $\eta = Cte$ are paraboloids of revolution around axis Oz.

Elliptical coordinates (ξ,μ,φ):

Based on the cylindrical representation, we write:

$\rho = \ell\,\sqrt{(\xi 2 - 1)(1-\mu 2)}$ and $z = \ell\,\xi\,\mu$ where ℓ is called the transformation parameter.
We obtain $W(\zeta,\eta,\varphi) = \dfrac{f(\varphi)}{(\xi^2-1)(1-\mu^2)} + \dfrac{g(\xi)}{\xi^2-\mu^2} + \dfrac{h(\mu)}{\xi^2-\mu^2}$.

The surfaces $\xi = Cte$ are a family of ellipsoids. The surfaces $\mu = Cte$ are a family of hyperboloids.

and a potential energy of the form

$$W = \frac{W_1 + \cdots + W_n}{U_1 + \cdots + U_n},$$

where U_i, V_i and W_i, with $i \in \{1, \cdots, n\}$, depend only on the variables q_i, where $i \in \{1, \cdots, n\}$.

The kinetic energy and potential energy are explicitly independent of time. Thus, t is a secondary parameter and $H = T + W = E$.

This is a case with separation of variables. The expressions for the conjugate variables p_i are

$$p_i = (U_1 + \cdots + U_n)\, V_i\, \dot{q}_i \text{ with } i \in \{1, \cdots, n\}.$$

We obtain

$$H = T + W = \frac{1}{2} \frac{\dfrac{p_1^2}{V_1} + \cdots + \dfrac{p_n^2}{V_n} + W_1 + \cdots + W_n}{U_1 + \cdots + U_n}.$$

We search for solutions of the Jacobi equation in the form

$$S = -E\, t + S_1(q_1) + \cdots + S_n(q_n). \qquad [5.10]$$

As t is a secondary parameter, the Jacobi equation becomes

$$\frac{\sum_{i=1}^{n} \left(\dfrac{S_i'^2}{V_i} + W_i \right)}{\sum_{i=1}^{n} U_i} = 2\, E,$$

or

$$\sum_{i=1}^{n} \frac{S_i'^2}{V_i} + W_i - 2\, E U_i = 0 \qquad [5.11]$$

and equation [5.11] is equivalent to

$$\frac{S_i'^2}{V_i} + W_i - 2\, E U_i = \beta_i \quad \text{where} \quad i \in \{1, \cdots, n\}, \quad \text{with} \quad \sum_{i=1}^{n} \beta_i = 0.$$

Consequently, $S_i' = \sqrt{V_i}\sqrt{(\beta_i + 2\,EU_i - W_i)}$ and by integration,

$$\begin{cases} S = -E\,t + \sum_{i=1}^{n} S_i\,(q_i) \\ \text{with} \\ S_i\,(q_i) = \int_{q_{i0}}^{q_i} \sqrt{V_i}\sqrt{(\beta_i + 2\,EU_i - W_i)}\,dq_i. \end{cases}$$

where $\beta_n = -\sum_{i=1}^{n-1} \beta_i$. Then,

$$t - t_0 = \sum_{i=1}^{n} \frac{\partial S_i}{\partial E}$$

represents the *law of time on the trajectory*.

The $n-1$ equations

$$\frac{\partial S}{\partial \beta i} + \alpha_i = 0 \quad \Longleftrightarrow \quad \frac{\partial S_i}{\partial \beta i} - \frac{\partial S_n}{\partial \beta n} + \alpha_i = 0 \text{ where } i \in \{1, \cdots, n-1\}$$

represent the *equations of the trajectory*.

5.5. A specific change of canonical variables

If the Hamiltonian is independent of time, we search for a change of the canonical variables

$$(\boldsymbol{P}, \boldsymbol{Q}, t) \longrightarrow (\boldsymbol{A}, \boldsymbol{B}, t)$$

brought about by a generating function of the form $F(\boldsymbol{Q}, \boldsymbol{A})$, which does not depend on time t, and such that the new Hamiltonian depends only on $\boldsymbol{A}$, i.e. $K = K(\boldsymbol{A})$. Equation [5.4]

$$\boldsymbol{P}^\star d\boldsymbol{Q} - H dt - (\boldsymbol{A}^\star d\boldsymbol{B} - K dt) = dS,$$

can be written as

$$\boldsymbol{P}^\star d\boldsymbol{Q} + \boldsymbol{B}^\star d\boldsymbol{A} + (K - H)\,dt = d(S + \boldsymbol{A}^\star \boldsymbol{B}).$$

Let us write $F = S + \boldsymbol{A}^\star \boldsymbol{B}$

$$\boldsymbol{P}^\star d\boldsymbol{Q} + \boldsymbol{B}^\star d\boldsymbol{A} + (K - H)\,dt = dF$$

and we deduce

$$\begin{cases} \boldsymbol{P}^\star = \dfrac{\partial F(\boldsymbol{Q},\boldsymbol{A})}{\partial \boldsymbol{Q}} \\ \boldsymbol{B}^\star = \dfrac{\partial F(\boldsymbol{Q},\boldsymbol{A})}{\partial \boldsymbol{A}} \\ K - H = \dfrac{\partial F(\boldsymbol{Q},\boldsymbol{A})}{\partial t} \end{cases}.$$

Since $\dfrac{\partial F(\boldsymbol{Q},\boldsymbol{A})}{\partial t} = 0$, we obtain $K(\boldsymbol{A}) = H(\boldsymbol{P},\boldsymbol{Q})$, and the new Hamilton equations are written as

$$\begin{cases} \dfrac{d\boldsymbol{B}}{dt} = \left(\dfrac{\partial K}{\partial \boldsymbol{A}}\right)^\star = K'^\star(\boldsymbol{A}) \\ \dfrac{d\boldsymbol{A}}{dt} = -\left(\dfrac{\partial K}{\partial \boldsymbol{B}}\right)^\star = \boldsymbol{0} \end{cases}.$$

We obtain

$$\boldsymbol{A} = \boldsymbol{A}_0 \quad \text{and} \quad \dot{\boldsymbol{B}} = K'^\star(\boldsymbol{A}_0) = \dot{\boldsymbol{B}}_0$$

where $\boldsymbol{A}_0$ and $\dot{\boldsymbol{B}}_0$ are constant vectors. Finally,

$$\boldsymbol{B} = \dot{\boldsymbol{B}}_0\, t + \boldsymbol{B}_0 \quad \text{and} \quad K(\boldsymbol{A}) = K(\boldsymbol{A}_0) = E \qquad [5.12]$$

where $\boldsymbol{B}_0$ is a constant vector and E is the total energy of the system. The equation that gives the generating function F is

$$H\left(\left(\frac{\partial F}{\partial \boldsymbol{Q}}\right)^\star, \boldsymbol{Q}\right) = E \qquad [5.13]$$

A complete solution of equation [5.13] must take into account not only the parameter E, which we denote by α_1, but also, according to Remark 5.1, the $n-1$ non-additive parameters $\alpha_2, \cdots, \alpha_n$. Thus, $\boldsymbol{A} = [\alpha_1, \cdots, \alpha_n]^\star$ and the solution of the partial differential equation [5.13] is in the form $F(\boldsymbol{Q},\boldsymbol{A})$. From equations [5.12] and [5.13], we deduce the equations of motion,

$$\dot{\boldsymbol{B}}_0\, t + \boldsymbol{B}_0 = \left(\frac{\partial F(\boldsymbol{Q},\boldsymbol{A}_0)}{\partial \boldsymbol{A}}\right)^\star.$$

5.6. Multi-periodic systems. Action variables

Let consider a Liouville dynamical system. We can choose the function F as a sum of functions depending only on a single parameter $q_i,\ i \in (1, \cdots, n)$,

$$F(\boldsymbol{Q}, \boldsymbol{A}) = F_1(q_1, \boldsymbol{A}) + \cdots + F_n(q_n, \boldsymbol{A}).$$

We assume that the system is multi-periodic. We have

$$\forall\, i \in \{1, \cdots, n\},\ \exists\, T_i \in R \quad \text{such that} \quad \forall\, t \in R, \quad q_i(t + T_i) = q_i(t) + 2\pi.$$

and we search for a new Hamiltonian in the form $K(\boldsymbol{A})$.

This is a case studied by Delaunay in his celestial mechanics' work (later developments of this work can be found in (Béletski 1986)). Let us write

$$J_i = \oint_{T_i} p_i\, dq_i = \oint_{T_i} \frac{\partial F_i}{\partial q_i}\, dq_i, \quad i \in \{1, \cdots, n\},$$

where $\oint_{T_i}$ denotes the integration on period T_i associated with the motion of parameter q_i. We write

$$\boldsymbol{J} = [J_1, \cdots, J_n]^\star.$$

With $\boldsymbol{J}$ substituting $\boldsymbol{A}$, we search for $\boldsymbol{B}$ and $K(\boldsymbol{J})$ such that

$$\boldsymbol{P}^\star\, d\boldsymbol{Q} + \boldsymbol{B}^\star\, d\boldsymbol{J} + [K(\boldsymbol{J}) - H(\boldsymbol{P}, \boldsymbol{Q})]\ dt = dF.$$

The new Hamilton equations are

$$\begin{cases} \dfrac{d\boldsymbol{B}}{dt} = \left(\dfrac{\partial K}{\partial \boldsymbol{J}}\right)^\star \equiv K'^\star(\boldsymbol{J}) \\ \dfrac{d\boldsymbol{J}}{dt} = -\left(\dfrac{\partial K}{\partial \boldsymbol{B}}\right)^\star = \boldsymbol{0} \end{cases} \qquad [5.14]$$

and the equations of motion are given by

$$\dot{\boldsymbol{B}}_0\, t + \boldsymbol{B}_0 = \left(\frac{\partial F\,(\boldsymbol{Q}, \boldsymbol{J}_0)}{\partial \boldsymbol{J}}\right)^\star$$

where $\boldsymbol{J} = \boldsymbol{J}_0$, with $\boldsymbol{B}_0$ and $\dot{\boldsymbol{B}}_0$ constant vectors.

The vector components J_i, $i \in \{1, \cdots, n\}$ are called *variables of action* because $\oint_{T_i} p_i \, dq_i$ has the dimension of $time \times energy$, i.e. the dimension of action.

The constants β_i, $i \in \{1, \cdots, n\}$, components of $\boldsymbol{B}$, are called *angular variables* because they have no physical dimension since F has the dimension of action, and consequently, $\boldsymbol{B}^\star = \dfrac{\partial F(\boldsymbol{Q}, \boldsymbol{J})}{\partial \boldsymbol{J}}$ is dimensionless.

Over a period T_i, we write $\Delta\beta_i = \oint_{T_i} \dfrac{\partial \beta_i}{\partial q_i} \, dq_i$. It can be deduced

$$\Delta\beta_i = \oint_{T_i} \frac{\partial^2 F}{\partial q_i \partial J_i} \, dq_i = \frac{\partial}{\partial J_i} \oint_{T_i} \frac{\partial F}{\partial q_i} \, dq_i = \frac{\partial}{\partial J_i} \oint_{T_i} p_i \, dq_i = \frac{\partial J_i}{\partial J_i} = 1.$$

Because of equation [5.14], $\dfrac{d\beta_i}{dt}$ is constant in motion and

$$\Delta\beta_i = \oint_{T_i} \frac{d\beta_i}{dt} \, dt = \frac{d\beta_i}{dt} \oint_{T_i} dt = \frac{d\beta_i}{dt} \, T_i.$$

Thus, $\dfrac{d\beta_i}{dt} \, T_i = 1$ and we denote

$$\nu_i = \frac{d\beta_i}{dt} = \frac{1}{T_i},$$

where ν_i represents the frequency relatively to parameter q_i, and equation [5.14] implies

$$\nu_i = \frac{\partial K}{\partial J_i}.$$

EXAMPLE 5.2.– ***Back to a one-dimensional harmonic oscillator***. This is the case considered in Example 5.1, where kinetic and potential energy are

$$T = \frac{1}{2} \, m \, \dot{q}^2 \quad \text{and} \quad W = \frac{1}{2} \, k \, q^2, \text{ with } k > 0.$$

The Hamiltonian is

$$H = \frac{1}{2m} \left(\frac{\partial F}{\partial q} \right)^2 + \frac{1}{2} \, k \, q^2 = E,$$

where F is the generating function and E is the total energy of the system. Since there is only one parameter, it exists as a single action variable denoted by J,

$$J = \oint_T p\,dq = \oint_T \frac{\partial F}{\partial q}\,dq = \sqrt{m\,k}\oint_T \sqrt{\frac{2E}{k} - q^2}\,dq.$$

By writing $q = \sqrt{\dfrac{2E}{k}}\,\sin\,\theta$, we obtain

$$J = \int_0^{2\pi} 2E\sqrt{\frac{m}{k}\cos^2\theta}\,d\theta = 2\pi\,E\sqrt{\frac{m}{k}}.$$

We deduce $K = E = \dfrac{J}{2\pi}\sqrt{\dfrac{k}{m}}$ and the frequency $\nu = \dfrac{\partial K}{\partial J} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$.

EXAMPLE 5.3.– ***The three-dimensional harmonic oscillator***. Consider a three-dimensional harmonic oscillator with unequal forces along the three orthogonal axes. The kinetic and potential energies of the system are

$$T = \frac{1}{2}\,m\,\left(\dot{q_1}^2 + \dot{q_2}^2 + \dot{q_3}^2\right) \quad \text{and} \quad W = \frac{1}{2}\left(k_1\,q_1^2 + k_2\,q_2^2 + k_3\,q_3^2\right),$$

where k_1, k_2, k_3 are positive constants. We get

$$H = \frac{1}{2}\left[\frac{{p_1}^2}{m} + k_1\,q_1^2\right] + \frac{1}{2}\left[\frac{{p_2}^2}{m} + k_2\,q_2^2\right] + \frac{1}{2}\left[\frac{{p_3}^2}{m} + k_3\,q_2^2\right]$$

and the generating function F satisfies

$$\left[\left(\frac{\partial F_1}{\partial q_1}\right)^2 + m\,k_1\,q_1^2\right] + \left[\left(\frac{\partial F_2}{\partial q_2}\right)^2 + m\,k_2\,q_2^2\right] + \left[\left(\frac{\partial F_3}{\partial q_3}\right)^2 + m\,k_3\,q_3^2\right] = 2m\,E,$$

which is a case of separation of variables; we can deduce

$$\left(\frac{\partial F_i}{\partial q_i}\right)^2 + m\,k_i\,q_i^2 = 2m\,\alpha_i \quad i \in \{1,2,3\} \quad \text{with} \quad \sum_1^3 \alpha_i = E.$$

We have

$$J_i = \oint_{T_i} p_i\,dq_i = \oint_{T_i} \frac{\partial F}{\partial q_i}\,dq_i = \oint_T \sqrt{2m\,\alpha_i - mk_i\,q_i^2}\,dq_i, \quad i \in \{1,2,3\}.$$

Using the same calculus as in Example 5.2, we obtain

$$J_i = 2\pi\alpha_i\sqrt{\frac{m}{k_i}}, \quad i \in \{1,2,3\}.$$

Hence,

$$J_1\sqrt{k_1} + J_2\sqrt{k_2} + J_3\sqrt{k_3} = 2\pi\sqrt{m}\,E = 2\pi\sqrt{m}\,K$$

and frequencies of the three oscillations on the three orthogonal axes are

$$\nu_i = \frac{\partial K}{\partial J_i} = \frac{1}{2\pi}\sqrt{\frac{k_i}{m}}, \quad i \in \{1,2,3\}.$$

6

Spaces of Mechanics – Poisson Brackets

6.1. Spaces in analytical mechanics

The calculations used in the previous chapters have led to different equivalent configurations:

a) The n-dimensions' *configuration space*. The space has parameters $\boldsymbol{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$, data $G\left(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t\right)$ and unknown $\boldsymbol{Q}(t)$. The motion is described by the Lagrange equations

$$\frac{d}{dt}\frac{\partial G}{\partial \dot{\boldsymbol{Q}}} - \frac{\partial G}{\partial \boldsymbol{Q}} = \boldsymbol{0}.$$

b) The $n+1$ dimensions' *configuration space time*. The space has parameters $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix}$, data $L\left(\boldsymbol{X}, \boldsymbol{X}'\right)$ and unknown $\boldsymbol{X}(\tau)$. The equations of motion are

$$\frac{d}{d\tau}\frac{\partial L}{\partial \boldsymbol{X}'} - \frac{\partial L}{\partial \boldsymbol{X}} = \boldsymbol{0}.$$

c) The $2n$ dimensions' *phase space*. The space has parameters $\boldsymbol{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ and $\boldsymbol{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$, data $H\left(\boldsymbol{P}, \boldsymbol{Q}, t\right)$ and unknowns $\boldsymbol{Q}(t), \boldsymbol{P}(t)$. The equations of motion

are the Hamilton equations

$$\begin{cases} \dot{\boldsymbol{Q}} = \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} \\ \dot{\boldsymbol{P}} = -\left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} \end{cases}.$$

d) The $2n+2$ dimensions' *homogeneous phase space*. The space has parameters $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$, where $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{Q} \\ t \end{bmatrix}$ and $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{P} \\ h \end{bmatrix}$. There exists a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ (or between $h, t, \boldsymbol{P}, \boldsymbol{Q}$). Thus, parameters belong to a hypersurface called the surface of states.

e) $2n+1$ dimensions' *state space*. The space has parameters $\boldsymbol{Z} = \boldsymbol{Z}(\tau)$, where $\boldsymbol{Z}$ satisfies relation $\varphi(\boldsymbol{Z}) = 0$ equivalent to $F(\boldsymbol{X}, \boldsymbol{Y}) = 0$ (or $h + H(\boldsymbol{P}, \boldsymbol{Q}, t) = 0$). The equations of motion are

$$\begin{cases} \dfrac{d\boldsymbol{X}}{d\tau} = \left(\dfrac{\partial F}{\partial \boldsymbol{Y}}\right)^{\star} \\ \dfrac{d\boldsymbol{Y}}{d\tau} = -\left(\dfrac{\partial F}{\partial \boldsymbol{X}}\right)^{\star} \end{cases}. \qquad [6.1]$$

If we denote $\varphi(\boldsymbol{Z}) \equiv F(\boldsymbol{X}, \boldsymbol{Y})$, thus $\dfrac{\partial \varphi}{\partial \boldsymbol{Z}} = \left[\dfrac{\partial F}{\partial \boldsymbol{X}}, \dfrac{\partial F}{\partial \boldsymbol{Y}}\right]$. The Hamilton equations in the state space – whose surface equation is $F(\boldsymbol{X}, \boldsymbol{Y}) = 0$ – can be written in the form of equation [6.1]

Let us write $\mathbf{0}_n$ for the $n \times n$ zero matrix and $\mathbf{1}_n$ for the $n \times n$ identity matrix. We denote

$$\dot{\mathbf{I}}_{2n} = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix}$$

therefore, $\dot{\mathbf{I}}_{2n}^{\star} = -\dot{\mathbf{I}}_{2n}$ and $\dot{\mathbf{I}}_{2n}^{\,2} = -\mathbf{1}_{2n}$.

The matrix $\dot{\mathbf{I}}_{2n}$ is called the $2n$-dimensional *symplectic matrix*. System [6.1] is the equivalent of

$$\frac{d\boldsymbol{Z}}{d\tau} = \dot{\mathbf{I}}_{2n+2}\left(\frac{\partial \varphi}{\partial \boldsymbol{Z}}\right)^{\star} \qquad [6.2]$$

Matrix $\dot{\mathbf{I}}_{2n+2}$ corresponds to the symplectic matrix of the $2n+2$ dimensions' homogeneous phase space. Let us also note that

$$\begin{bmatrix} \dfrac{\boldsymbol{dQ}}{dt} \\ \dfrac{\boldsymbol{dP}}{dt} \end{bmatrix} = \dot{\mathbf{I}}_{2n} \begin{bmatrix} \left(\dfrac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} \\ \left(\dfrac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} \end{bmatrix}.$$

NOTATION.– We denote $\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{Y}_1 \end{bmatrix}$ and $\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{X}_2 \\ \boldsymbol{Y}2 \end{bmatrix}$, then

$$\boldsymbol{Z}_1^{\star}\, \dot{\mathbf{I}}_{2n+2}\, \boldsymbol{Z}_2 = [\boldsymbol{X}_1^{\star}, \boldsymbol{Y}_1^{\star}] \begin{bmatrix} \mathbf{0}_{n+1} & \mathbf{1}_{n+1} \\ -\mathbf{1}_{n+1} & \mathbf{0}_{n+1} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_2 \\ \boldsymbol{Y}2 \end{bmatrix}.$$

In the absence of ambiguity, we remove the indices $n, n+1, 2n, 2n+2$ for respective phase spaces. We deduce

$$\boldsymbol{Z}_1^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z}_2 = \boldsymbol{X}_1^{\star}\boldsymbol{Y}_2 - \boldsymbol{Y}_1^{\star}\boldsymbol{X}_2 = -\boldsymbol{Z}_2^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z}_1.$$

DEFINITION 6.1.– Term $\boldsymbol{Z}_1^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z}_2$ is called the ***symplectic scalar product of vectors*** $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ defined on a vector space with an even dimension.

Expression $\boldsymbol{Z}_1^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z}_2$ is the symplectic scalar product in the basis of $\mathcal{E}^{2n+2}$. We also write

$$\sigma\left(\boldsymbol{Z}_1\right)\left(\boldsymbol{Z}_2\right) = \boldsymbol{Z}_1^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z}_2,$$

and for any couple of vectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, we get

$$\sigma\left(\boldsymbol{V}_1\right)\left(\boldsymbol{V}_2\right) = -\sigma\left(\boldsymbol{V}_2\right)\left(\boldsymbol{V}_1\right).$$

REMARK 6.1.– For any vector $\boldsymbol{Z}$ such that $\boldsymbol{Z}^{\star}\, \dot{\mathbf{I}}\, \boldsymbol{Z} = 0$, we deduce

$$\frac{\partial \varphi}{\partial \boldsymbol{Z}}\, \dot{\mathbf{I}} \left(\frac{\partial \varphi}{\partial \boldsymbol{Z}}\right)^{\star} = 0 \quad \text{which can be written} \quad \frac{\partial \varphi}{\partial \boldsymbol{Z}} \frac{d\boldsymbol{Z}}{d\tau} = 0.$$

Vector $\left(\dfrac{\partial \varphi}{\partial \boldsymbol{Z}}\right)^{\star}$ represents $\mathrm{grad}\varphi$ in the $\mathcal{E}^{2n+2}$ space. Vector $\dot{\mathbf{I}}\ \mathrm{grad}\varphi$ is normal to $\mathrm{grad}\varphi$ and is a tangent vector to the trajectory of the motion in the state space (see Figure 6.1).

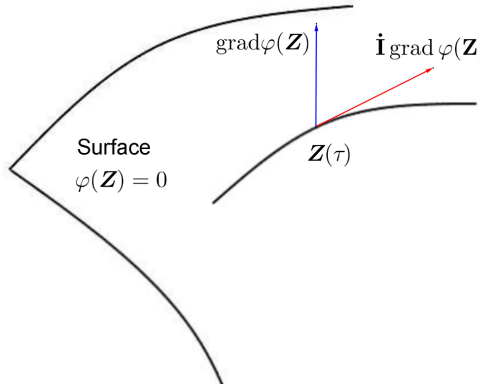


Figure 6.1. *Trajectory of Z in the space state. The vector $\dot{\mathbf{I}}$ gradφ is tangent to the trajectory represented in the state space. For a color version of this figure, see www.iste.co.uk/gouin/mechanics.zip*

6.2. Dynamical variables – Poisson brackets

A function of the state of system $(\mathcal{S})$ is called a dynamical variable. In the various configurations, a dynamical variable u is a function of the parameters:

a) in the phase space, $u = u(\boldsymbol{P}, \boldsymbol{Q}, t)$;

b) in the homogeneous phase space, $u = u(\boldsymbol{P}, \boldsymbol{Q}, h, t) \equiv u(\boldsymbol{X}, \boldsymbol{Y})$;

c) in the state space, $u = u(\boldsymbol{Z})$.

6.2.1. *Evolution equation of a dynamical variable*

The dynamical variable is assumed to be differentiable.

$$\texttt{Case}\ (a): \quad u = u\left(\boldsymbol{P}, \boldsymbol{Q}, t\right) \quad \Longrightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \boldsymbol{P}} \frac{d\boldsymbol{P}}{dt} + \frac{\partial u}{\partial \boldsymbol{Q}} \frac{d\boldsymbol{Q}}{dt}.$$

Term $\dfrac{du}{dt}$ denotes the material derivative or the derivative along the motion of system $(\mathcal{S})$. We obtain

$$\frac{du}{dt} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial \boldsymbol{P}} \left(\frac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star} + \frac{\partial u}{\partial \boldsymbol{Q}} \left(\frac{\partial H}{\partial \boldsymbol{P}}\right)^{\star}.$$

$$\texttt{Case}\ (b): \quad u = u(\boldsymbol{X}, \boldsymbol{Y}) \quad \Longrightarrow \quad \frac{du}{d\tau} = \frac{\partial u}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\tau} + \frac{\partial u}{\partial \boldsymbol{Y}} \frac{d\boldsymbol{Y}}{d\tau}.$$

Along the motion of system $(\mathcal{S})$, we deduce

$$\frac{du}{d\tau} = \frac{\partial u}{\partial \boldsymbol{X}} \left(\frac{\partial F}{\partial \boldsymbol{Y}}\right)^{\star} - \frac{\partial u}{\partial \boldsymbol{Y}} \left(\frac{\partial F}{\partial \boldsymbol{X}}\right)^{\star} \Longleftrightarrow$$

$$\frac{du}{d\tau} = \left[\frac{\partial u}{\partial \boldsymbol{X}}, \frac{\partial u}{\partial \boldsymbol{Y}}\right] \dot{\mathbf{I}} \begin{bmatrix} \left(\dfrac{\partial F}{\partial \boldsymbol{X}}\right)^{\star} \\ \left(\dfrac{\partial F}{\partial \boldsymbol{Y}}\right)^{\star} \end{bmatrix}.$$

DEFINITION 6.2.– The scalar

$$[u, H]_{(\boldsymbol{Q},\boldsymbol{P})} = \frac{\partial u}{\partial \boldsymbol{Q}} \left(\frac{\partial H}{\partial \boldsymbol{P}}\right)^{\star} - \frac{\partial u}{\partial \boldsymbol{P}} \left(\frac{\partial H}{\partial \boldsymbol{Q}}\right)^{\star}$$

is called the ***Poisson bracket*** of u and H relatively to variables $\boldsymbol{Q}$ and $\boldsymbol{P}$. Its expanded form is

$$[u, H]_{(\boldsymbol{Q},\boldsymbol{P})} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i}\right).$$

Consequently, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + [u, H]_{(\boldsymbol{Q},\boldsymbol{P})}.$$

By denoting $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{P} \end{bmatrix}$, we obtain

$$[u, H]_{(\boldsymbol{Q},\boldsymbol{P})} = \frac{\partial u}{\partial \boldsymbol{R}} \dot{\mathbf{I}} \left(\frac{\partial H}{\partial \boldsymbol{R}}\right)^{\star}.$$

$$\texttt{Case}(c): \qquad \frac{du}{d\tau} = \frac{\partial u}{\partial \boldsymbol{Z}} \dot{\mathbf{I}} \left(\frac{\partial \varphi}{\partial \boldsymbol{Z}}\right)^{\star} = [u, F]_{(\boldsymbol{X},\boldsymbol{Y})},$$

where depending on the cases, $\dot{\mathbf{I}}$ has dimension $2n$ or $2n + 2$. We can also write

$$\frac{du}{d\tau} = \frac{\partial u}{\partial \boldsymbol{Z}} \frac{d\boldsymbol{Z}}{d\tau}.$$

REMARK 6.2.– We conclude

$$\dot{q}_i = [q_i, H]_{(\boldsymbol{Q},\boldsymbol{P})} \quad \text{and} \quad \dot{p}_i = [p_i, H]_{(\boldsymbol{Q},\boldsymbol{P})}$$

6.2.2. *First integral*

DEFINITION 6.3.– A dynamical variable u is a ***first integral*** if and only if u is constant along each trajectory of the motion.

REMARK 6.3.– The first integral u implies $\dfrac{du}{dt} \equiv [u, F]_{(\boldsymbol{X},\boldsymbol{Y})} = 0$. In the case where $\dfrac{\partial u}{\partial t} = 0$, u is a first integral if and only if $[u, H]_{(\boldsymbol{Q},\boldsymbol{P})} = 0$.

6.3. Poisson bracket of two dynamical variables

DEFINITION 6.4.– Given two differentiable dynamical variables denoted by u and v:

$$(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{R}^{2n} \longrightarrow u\,(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{R} \quad \text{and} \quad (\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{R}^{2n} \longrightarrow v\,(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{R},$$

the scalar denoted by $[u, v]_{(\boldsymbol{X},\boldsymbol{Y})}$ and defined by

$$[u, v]_{(\boldsymbol{X},\boldsymbol{Y})} = \frac{\partial u}{\partial \boldsymbol{X}} \left(\frac{\partial v}{\partial \boldsymbol{Y}}\right)^{\star} - \frac{\partial u}{\partial \boldsymbol{Y}} \left(\frac{\partial v}{\partial \boldsymbol{X}}\right)^{\star}$$

$$\text{where} \quad \begin{cases} \dfrac{\partial u}{\partial \boldsymbol{X}} = \left[\dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_n}\right] \quad \text{and} \quad \dfrac{\partial u}{\partial \boldsymbol{Y}} = \left[\dfrac{\partial u}{\partial y_1}, \cdots, \dfrac{\partial u}{\partial y_n}\right] \\ \dfrac{\partial v}{\partial \boldsymbol{X}} = \left[\dfrac{\partial v}{\partial x_1}, \cdots, \dfrac{\partial v}{\partial x_n}\right] \quad \text{and} \quad \dfrac{\partial v}{\partial \boldsymbol{Y}} = \left[\dfrac{\partial v}{\partial y_1}, \cdots, \dfrac{\partial v}{\partial y_n}\right] \end{cases}$$

is called the Poisson bracket of u and v relatively to $\boldsymbol{X}$ and $\boldsymbol{Y}$.

In expanded form, it is

$$[u, v]_{(\boldsymbol{X},\boldsymbol{Y})} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial y_i} - \frac{\partial u}{\partial y_i}\frac{\partial v}{\partial x_i}\right) \quad \text{where} \quad \boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

In the following, the Poisson bracket is denoted by $[u, v]$ without referring to variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, where $(\boldsymbol{X}, \boldsymbol{Y})$ denotes a point in the phase space. The results are identical for the Poisson brackets taken relatively to the couple $(\boldsymbol{P}, \boldsymbol{Q})$. The relations obtained are not relevant to the study of material systems.

In the mathematical case, we say *Lie brackets*. The Lie brackets constitute the *Lie algebra*.

6.3.1. *Properties of Poisson brackets*

THEOREM 6.1.– A Poisson bracket (or a Lie bracket) is an antisymmetric bilinear mapping.

– The Poisson bracket is bilinear

For u_1, u_2, v mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}$ and for real λ,

$$[u_1 + u_2, v] = [u_1, v] + [u_2, v] \quad \text{and} \quad [\lambda u, v] = \lambda\,[u, v]\,.$$

For u, v_1, v_2 mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}$ and for real λ,

$$[u, v_1 + v_2] = [u, v_1] + [u, v_2] \quad \text{and} \quad [u, \lambda v] = \lambda\,[u, v]\,.$$

– The Poisson bracket is antisymmetric

$[u, v] = -\,[v, u]$. Consequently, $[u, u] = 0$.

REMARK 6.4.– For all dynamical variables u, v, w, we have

$$[uv, w] = u\,[v, w] + v\,[u, w]\,.$$

This property immediately results from the properties of partial derivatives associated with the dynamical variables u, v, w. Furthermore, for $\dfrac{\partial}{\partial \alpha}$ denoting a partial derivative associated with one of variables defining u and v,

$$\frac{\partial}{\partial \alpha}\,[u, v] = \left[\frac{\partial u}{\partial \alpha}, v\right] + \left[u, \frac{\partial v}{\partial \alpha}\right].$$

DEFINITION 6.5.– ***Derivation***. A mapping d from $\mathcal{A}\left(\mathbb{R}^{2n}, \mathbb{R}\right)$ to $\mathcal{A}\left(\mathbb{R}^{2n}, \mathbb{R}\right)$ is a derivation if and only if it satisfies the three following properties:

$$d\,(u + v) = du + dv,$$
$$d\,(uv) = du\,v + u\,dv,$$

u being a constant mapping, $du = 0$.

THEOREM 6.2.– ***The Jacobi identity***. Given three twice-differentiable mappings, u, v, w of $\mathcal{A}\left(\mathbb{R}^{2n}, \mathbb{R}\right)$, we have the relation

$$\big[\,[u, v]\,, w\big] + \big[\,[v, w]\,, u\big] + \big[\,[w, u]\,, v\big] = 0.$$

PROOF.– (Proof for the Jacobi identity)

a) Let us write $\mathcal{D}_u v = [u, v]$. Then, $\mathcal{D}_u$ is a derivation. Indeed,

$$\mathcal{D}_u\,(v + w) = [u, v + w] = [u, v] + [u, w] = \mathcal{D}_u v + \mathcal{D}_u w,$$
$$\mathcal{D}_u(v\,w) = [u, v\,w] = v[u, w] + w[u, v] = v\,\mathcal{D}_u w + w\mathcal{D}_u v,$$
$$if\ \ v = a \text{ with } a \text{ constant, then } \mathcal{D}_u a = [u, a] = 0.$$

b) Let d denote the derivation $\mathcal{D}_w$, then

$$d[u, v] = [du, v] + [u, dv].$$

Indeed,

$$[u, v] = \frac{\partial u}{\partial \boldsymbol{X}}\left(\frac{\partial v}{\partial \boldsymbol{Y}}\right)^{\star} - \frac{\partial u}{\partial \boldsymbol{Y}}\left(\frac{\partial v}{\partial \boldsymbol{X}}\right)^{\star}$$

and we apply d to $[u, v]$.

c) $\mathcal{D}_w[u, v] = \big[w, [u, v]\big]$. Furthermore, according to *(b)*,

$$\mathcal{D}_w[u, v] = [\mathcal{D}_w u, v] + [u, \mathcal{D}_w v] = \big[[\,w, u], v\big] + \big[u, [w, v]\big].$$

But, the Poisson bracket is antisymmetric, then,

$$-\big[[\,u, v], w\big] = \big[[\,w, u], v\big] + \big[[\,v, w], u\big],$$

or

$$\big[[\,u, v], w\big] + \big[[\,v, w], u\big] + \big[[\,w, u], v\big] = 0,$$

which demonstrates the Jacobi identity. ■

DEFINITION 6.6.– ***Lie algebra***. Since Poisson brackets (or Lie brackets) are antisymmetric bilinear mappings satisfying the Jacobi identity, the dynamical variables have a Lie algebra structure.

REMARK 6.5.– The Jacobi relation can be interpreted in the form

$$\mathcal{D}_{[u,v]}w = \mathcal{D}_u\mathcal{D}_v w - \mathcal{D}_v\mathcal{D}_u w,$$

indeed,

$$\big[[\,u,v],w\big] = \big[u,[v,w]\big] - \big[v,[u,w]\big].$$

We deduce that

$$\mathcal{D}_u\mathcal{D}_v - \mathcal{D}_v\mathcal{D}_u$$

is a derivation and the Poisson bracket of two derivations is a derivation.

THEOREM 6.3.– The non-zero Poisson bracket of two first integrals is a first integral.

This is a consequence of the Jacobi identity. Indeed,

u is a first integral equivalent to $[u,F]=0$,

v is a first integral equivalent to $[v,F]=0$. We deduce

$$\big[[\,u,v],F\big] = \big[u,[v,F]\big] - \big[v,[u,F]\big] \;= 0.$$

Thus, $\big[[u,v],F\big]=0$; if $[u,v]\neq 0$, then $[u,v]$ constitutes a first integral.

6.3.2. *Application to the Noether theorem*

Let $\{\mathcal{T}_\alpha,\ \alpha\in I_1\subset\mathbb{R}\}$ be a Lie group whose infinitesimal operator is the vector field $\boldsymbol{\Phi}(\boldsymbol{X})$ defined on the configuration space $\mathcal{E}^{n+1}$. It is assumed the integral $a=\int_{(\mathcal{C})}\boldsymbol{Y}^\star(\boldsymbol{X})\,d\boldsymbol{X}$ is invariant by the Lie group. Along the curves $(\mathcal{C})$ where a is extremal, $u = \boldsymbol{Y}^\star\,\boldsymbol{\Phi}\,(\boldsymbol{X})$ is constant and u is a first integral. The differential equation associated with $\{\mathcal{T}_\alpha,\ \alpha\in I_1\subset\mathbb{R}\}$ is

$$\frac{d\boldsymbol{X}}{d\alpha} = \boldsymbol{\Phi}\,(\boldsymbol{X})\,.$$

Similarly, let $\{\mathcal{T}_\beta,\ \beta\in I_2\subset\mathbb{R}\}$ be a Lie group with the infinitesimal operator $\boldsymbol{\Psi}(\boldsymbol{X})$ making the integral a invariant. The differential equation associated with $\{\mathcal{T}_\beta,\ \beta\in I_2\subset\mathbb{R}\}$ is

$$\frac{d\boldsymbol{X}}{d\beta} = \boldsymbol{\Psi}\,(\boldsymbol{X})\,.$$

Along each curve $(\mathcal{C})$, $v = \boldsymbol{Y}^\star \boldsymbol{\Psi}(\boldsymbol{X})$ is a first integral. According to Theorem 6.3, $[u, v]$ is a first integral of the extremal curves of a. Furthermore,

$$\frac{\partial u}{\partial \boldsymbol{X}} = \boldsymbol{Y}^\star \boldsymbol{\Phi}'(\boldsymbol{X}), \ \frac{\partial u}{\partial \boldsymbol{Y}} = \boldsymbol{\Phi}^\star(\boldsymbol{X}), \quad \frac{\partial v}{\partial \boldsymbol{X}} = \boldsymbol{Y}^\star \boldsymbol{\Psi}'(\boldsymbol{X}), \ \frac{\partial v}{\partial \boldsymbol{Y}} = \boldsymbol{\Psi}^\star(\boldsymbol{X})$$

that yields

$$[u, v] = \frac{\partial u}{\partial \boldsymbol{X}} \left(\frac{\partial v}{\partial \boldsymbol{Y}}\right)^\star - \frac{\partial v}{\partial \boldsymbol{X}} \left(\frac{\partial u}{\partial \boldsymbol{Y}}\right)^\star = \boldsymbol{Y}^\star \left[\boldsymbol{\Phi}'(\boldsymbol{X})\, \boldsymbol{\Psi}(\boldsymbol{X}) - \boldsymbol{\Psi}'(\boldsymbol{X})\, \boldsymbol{\Phi}(\boldsymbol{X})\right]$$

where $\boldsymbol{\Phi}'(\boldsymbol{X}) = \dfrac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{X}}$ is represented by the Jacobian matrix $\left\{\dfrac{\partial \phi_i}{\partial X_j}\right\}$ with $i, j \in \{1, \cdots, n+1\}$. This is the same for $\boldsymbol{\Psi}'(\boldsymbol{X})$. Let us write

$$\boldsymbol{\Xi}(\boldsymbol{X}) = \boldsymbol{\Phi}'(\boldsymbol{X})\, \boldsymbol{\Psi}(\boldsymbol{X}) - \boldsymbol{\Psi}'(\boldsymbol{X})\, \boldsymbol{\Phi}(\boldsymbol{X}).$$

Thus, $[u, v] = \boldsymbol{Y}^\star \boldsymbol{\Xi}(\boldsymbol{X})$ and the vector field $\boldsymbol{\Xi}(\boldsymbol{X})$ is the infinitesimal operator of the Lie group associated with the differential equation $\dfrac{d\boldsymbol{X}}{d\gamma} = \boldsymbol{\Xi}(\boldsymbol{X})$. This group is denoted $\{\mathcal{T}_\gamma,\ \gamma \in I \subset \mathbb{R}\}$.

EXAMPLE 6.1.– Let $\{\mathcal{T}_\alpha,\ \alpha \in I_1 \subset \mathbb{R}\}$ be the group of rotations around axis Oz of the orthonormal frame $Oxyz$. In Cartesian coordinates

$$\boldsymbol{\Phi}(\boldsymbol{X}) = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}.$$

Let $\{\mathcal{T}_\beta,\ \beta \in I_2 \subset \mathbb{R}\}$ be the group of rotations around axis Oy of the orthonormal frame $Oxyz$. In Cartesian coordinates

$$\boldsymbol{\Psi}(\boldsymbol{X}) = \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}.$$

We deduce

$$\boldsymbol{\Phi}'(\boldsymbol{X}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Psi}'(\boldsymbol{X}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Hence,

$$\boldsymbol{\Xi}\left(\boldsymbol{X}\right)=\boldsymbol{\Phi}'\left(\boldsymbol{X}\right)\boldsymbol{\Psi}(\boldsymbol{X})-\boldsymbol{\Psi}'\left(\boldsymbol{X}\right)\boldsymbol{\Phi}(\boldsymbol{X})=\begin{bmatrix}0\\ z\\ -y\end{bmatrix}.$$

This is the infinitesimal operator associated with the group of rotations around axis Ox. Similarly, if invariance of rotation around axis Ox and invariance of rotation around axis Oy correspond to two first integrals, we can deduce a first integral associated with the group of rotations around axis Oz.

Let us note that the Poisson bracket of two first integrals does not always imply a new first integral

EXAMPLE 6.2.– Let $\{\mathcal{T}_\alpha,\ \alpha\in I_1\subset\mathbb{R}\}$ be the group of translations in direction Ox of orthonormal frame $Oxyz$. Then, in Cartesian coordinates,

$$\boldsymbol{\Phi}(\boldsymbol{X})=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\quad\text{and}\quad\boldsymbol{\Phi}'(\boldsymbol{X})=\mathbf{0}.$$

Let $\{\mathcal{T}_\beta,\ \beta\in I_2\subset\mathbb{R}\}$ be the group of translations in direction Oy of orthonormal frame $Oxyz$. Then, in Cartesian coordinates,

$$\boldsymbol{\Psi}(\boldsymbol{X})=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\quad\text{and}\quad\boldsymbol{\Psi}'(\boldsymbol{X})=\mathbf{0}.$$

Hence,

$$\Xi\left(\boldsymbol{X}\right)=\boldsymbol{\Phi}'\left(\boldsymbol{X}\right)\boldsymbol{\Psi}(\boldsymbol{X})-\boldsymbol{\Psi}'\left(\boldsymbol{X}\right)\boldsymbol{\Phi}(\boldsymbol{X})=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

The invariance by translation with respect to Ox and the invariance by translation with respect to Oy do not imply an additional first integral.

6.4. Canonical transformations

In the homogeneous phase space, the equations of motion are in the form [6.2],

$$\frac{d\boldsymbol{Z}}{d\tau}=\dot{\mathbf{I}}\left(\frac{\partial\varphi}{\partial\boldsymbol{Z}}\right)^{\star},$$

where $\dot{\mathbf{I}}$ denotes $\dot{\mathbf{I}}_{2n+2}$ and $\varphi(\boldsymbol{Z}) = 0$ denotes the constraint $h + H(\boldsymbol{P}, \boldsymbol{Q}, t) = 0$.

Consider the change of variables

$$\boldsymbol{Z} \in \mathcal{E}^{2n+2} \longrightarrow \tilde{\boldsymbol{Z}} = \psi(\boldsymbol{Z}) \in \mathcal{E}^{2n+2}.$$

Condition $\varphi(\boldsymbol{Z}) = 0$ is written in new variables

$$\tilde{\varphi}\left(\tilde{\boldsymbol{Z}}\right) \equiv \varphi\left(\psi^{-1}\left(\tilde{\boldsymbol{Z}}\right)\right) = 0.$$

The linear mapping $\dfrac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} = \dfrac{\partial \psi(\boldsymbol{Z})}{\partial \boldsymbol{Z}}$ is locally invertible. In new variables, we can write

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \tau} = \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} \frac{\partial \boldsymbol{Z}}{\partial \tau}$$

and, taking into account [6.2] and $\dfrac{\partial \varphi}{\partial \boldsymbol{Z}} = \dfrac{\partial \tilde{\varphi}}{\partial \tilde{\boldsymbol{Z}}} \dfrac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}$, we obtain

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \tau} = \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} \dot{\mathbf{I}} \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} \left(\frac{\partial \tilde{\varphi}}{\partial \tilde{\boldsymbol{Z}}}\right)^{\star}.$$

Let us denote

$$\mathbf{P} = \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} \dot{\mathbf{I}} \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star}$$

We call $\mathbf{P}$ the *Poisson matrix of the change of variables*. The change of variables is canonical if and only if

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{\tau}} = \dot{\mathbf{I}} \left(\frac{\partial \tilde{\varphi}}{\partial \tilde{\boldsymbol{Z}}}\right)^{\star}.$$

Hence,

THEOREM 6.4.– The change of variables,

$$\boldsymbol{Z} \in \mathcal{E}^{2n+2} \longrightarrow \tilde{\boldsymbol{Z}} = \psi(\boldsymbol{Z}) \in \mathcal{E}^{2n+2}$$

is canonical if and only if $\mathbf{P} = \dot{\mathbf{I}}$.

6.4.1. *Calculus of the Poisson matrix*

$$\text{Since } \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \text{ and } \tilde{\boldsymbol{Z}} = \begin{bmatrix} \tilde{\boldsymbol{X}} \\ \tilde{\boldsymbol{Y}} \end{bmatrix}, \text{ then } \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} = \begin{bmatrix} \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} & \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \\ \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} & \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \end{bmatrix}$$

and

$$\left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} = \begin{bmatrix} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\right)^{\star} & \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}}\right)^{\star} \\ \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\right)^{\star} & \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}}\right)^{\star} \end{bmatrix}.$$

We obtain

$$\begin{aligned} &\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} \,\dot{\mathbf{1}}\, \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} \\ &= \begin{bmatrix} \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} & \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \\ \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} & \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\right)^{\star} & \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}}\right)^{\star} \\ \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\right)^{\star} & \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}}\right)^{\star} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\right)^{\star} & \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}}\right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}}\right)^{\star} \\ \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\right)^{\star} & \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}}\right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}}\right)^{\star} \end{bmatrix}. \end{aligned}$$

The change of variables is canonical if and only if

$$\begin{cases} \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \right)^{\star} = 0, \\ \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \right)^{\star} = \mathbf{1}, \\ \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \right)^{\star} - \dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \left(\dfrac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \right)^{\star} = 0. \end{cases}$$

Let us note that

$$\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\frac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \right)^{\star} - \frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \left(\frac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \right)^{\star} = \mathbf{1},$$

is equivalent to

$$\frac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{X}} \left(\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \right)^{\star} - \frac{\partial \tilde{\boldsymbol{Y}}}{\partial \boldsymbol{Y}} \left(\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \right)^{\star} = -\mathbf{1}.$$

For a calculus in coordinates, we obtain results that depend on the chosen basis,

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix}, \ \tilde{\boldsymbol{X}} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{n+1} \end{bmatrix}, \ \boldsymbol{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n+1} \end{bmatrix} \text{ and } \tilde{\boldsymbol{Y}} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_{n+1} \end{bmatrix}.$$

$\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} = \left\{ \dfrac{\partial \tilde{x}_i}{\partial x_h} \right\}$ where i is the row index and h is the column index,

$\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} = \left\{ \dfrac{\partial \tilde{x}_j}{\partial y_h} \right\}$ where j is the row index and h is the column index,

$\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}} \left(\dfrac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}} \right)^{\star} = \{a_{ij}\}$ where i is the row index and j is the column index, with $a_{ij} = \displaystyle\sum_{h=1}^{n+1} \frac{\partial \tilde{x}_i}{\partial x_h} \frac{\partial \tilde{x}_j}{\partial y_h}$.

Thus,

$$\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\left(\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\right)^{\star}-\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{Y}}\left(\frac{\partial \tilde{\boldsymbol{X}}}{\partial \boldsymbol{X}}\right)^{\star}$$

has for generic element,

$$\sum_{h=1}^{n+1}\left(\frac{\partial \tilde{x}_i}{\partial x_h}\frac{\partial \tilde{x}_j}{\partial y_h}-\frac{\partial \tilde{x}_i}{\partial y_h}\frac{\partial \tilde{x}_j}{\partial x_h}\right)$$

where i is the row index and j is the column index, which is the Poisson bracket $[\tilde{x}_i,\tilde{x}_j]_{(\boldsymbol{X},\boldsymbol{Y})}$. Thus, for $i,j\in\{1,\cdots,n+1\}$, $[\tilde{x}_i,\tilde{x}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=0$. Similarly, for $i,j\in\{1,\cdots,n+1\}$, $[\tilde{y}_i,\tilde{y}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=0$. Finally, for $i,j\in\{1,\cdots,n+1\}$, $[\tilde{x}_i,\tilde{y}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=\delta_{ij}$, where $\delta_{ij}=0$ for $i\neq j$ and $\delta_{ij}=1$ for $i=j$.

COROLLARY 6.1.– In coordinate lines, the change of variables

$$\boldsymbol{Z}\in\mathcal{E}^{2n+2}\;\longrightarrow\;\tilde{\boldsymbol{Z}}=\psi\left(\boldsymbol{Z}\right)\in\mathcal{E}^{2n+2}.$$

is canonical if and only if for all $i,j\in\{1,\cdots,n+1\}$,

$$[\tilde{x}_i,\tilde{x}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=0,\quad [\tilde{y}_i,\tilde{y}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=0,\quad [\tilde{x}_i,\tilde{y}_j]_{(\boldsymbol{X},\boldsymbol{Y})}=\delta_{ij}.$$

EXAMPLE 6.3.– Determine the values of α and β such that the change of variables

$$\tilde{x}=x^{\alpha}\cos\beta y\;\text{ and }\;\tilde{y}=x^{\alpha}\sin\beta y$$

is canonical.

The only condition to be satisfied is $[\tilde{x},\tilde{y}]_{(\boldsymbol{x},\boldsymbol{y})}=1$. That leads to

$$\begin{aligned}\frac{\partial \tilde{x}}{\partial x}\frac{\partial \tilde{y}}{\partial y}-\frac{\partial \tilde{x}}{\partial y}\frac{\partial \tilde{y}}{\partial x}&=\left(\alpha x^{\alpha-1}\cos\beta y\right)\left(\beta x^{\alpha}\cos\beta y\right)\\&\quad+\left(\beta x^{\alpha}\sin\beta y\right)\left(\alpha x^{\alpha-1}\sin\beta y\right)=1,\end{aligned}$$

or,

$$\alpha\,\beta\,x^{2\alpha-1}=1.$$

The condition must be satisfied for all values of x. Hence,

$$2\alpha-1=0\;\text{ and }\;\alpha\,\beta=1\quad\Longleftrightarrow\quad\alpha=\frac{1}{2}\;\text{ and }\;\beta=2.$$

Finally,

$$\tilde{x} = \sqrt{x}\cos 2y \ \text{ and } \ \tilde{y} = \sqrt{x}\sin 2y.$$

REMARK 6.6.– Let us use the previous results to verify that the change of variables associated with a generating function of Jacobi is canonical. We have written that the extremals of $a = \int_{(C)} \boldsymbol{Y}^\star d\boldsymbol{X}$ where $F(\boldsymbol{X}, \boldsymbol{Y}) = 0$ are identical to the extremals of $b = \int_{(C)} \tilde{\boldsymbol{Y}}^\star d\tilde{\boldsymbol{X}}$ where $\tilde{F}\left(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}\right) = 0$.

PROOF.– For the proof, it is enough that there exists a generating function of Jacobi S such that

$$dS\left(\boldsymbol{X}, \tilde{\boldsymbol{X}}\right) = \boldsymbol{Y}^\star d\boldsymbol{X} - \tilde{\boldsymbol{Y}}^\star d\tilde{\boldsymbol{X}} \ \text{ with } \ F(\boldsymbol{X}, \boldsymbol{Y}) = 0.$$

We deduce

$$\boldsymbol{Y} = \left(\frac{\partial S\left(\boldsymbol{X}, \tilde{\boldsymbol{X}}\right)}{\partial \boldsymbol{X}}\right)^\star \text{ and } \tilde{\boldsymbol{Y}} = -\left(\frac{\partial S\left(\boldsymbol{X}, \tilde{\boldsymbol{X}}\right)}{\partial \tilde{\boldsymbol{X}}}\right)^\star \text{ with } F\left(\boldsymbol{X}, \left(\frac{\partial S}{\partial \boldsymbol{X}}\right)^\star\right) = 0.$$

Therefore,

$$F\left(\boldsymbol{X}, \left(\frac{\partial S\left(\boldsymbol{X}, \tilde{\boldsymbol{X}}\right)}{\partial \boldsymbol{X}}\right)^\star\right) = 0$$

represents the Jacobi partial differential equation in homogeneous variables.

Let us calculate $\dfrac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\, \dot{\mathbf{I}}\, \left(\dfrac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^\star$. We have

$$\begin{cases} d\boldsymbol{Y} = \dfrac{\partial}{\partial \boldsymbol{X}}\left(\dfrac{\partial S}{\partial \boldsymbol{X}}\right)^\star d\boldsymbol{X} + \dfrac{\partial}{\partial \tilde{\boldsymbol{X}}}\left(\dfrac{\partial S}{\partial \boldsymbol{X}}\right)^\star d\tilde{\boldsymbol{X}} \\ d\tilde{\boldsymbol{Y}} = -\dfrac{\partial}{\partial \boldsymbol{X}}\left(\dfrac{\partial S}{\partial \tilde{\boldsymbol{X}}}\right)^\star d\boldsymbol{X} - \dfrac{\partial}{\partial \tilde{\boldsymbol{X}}}\left(\dfrac{\partial S}{\partial \tilde{\boldsymbol{X}}}\right)^\star d\tilde{\boldsymbol{X}}, \end{cases}$$

which can be written

$$\begin{cases} d\boldsymbol{Y} = \mathbf{A}\, d\boldsymbol{X} + \mathbf{B}\, d\tilde{\boldsymbol{X}} \\ d\tilde{\boldsymbol{Y}} = -\mathbf{B}^\star d\boldsymbol{X} - \mathbf{C}\, d\tilde{\boldsymbol{X}} \end{cases} \text{ with } \ \mathbf{A} = \mathbf{A}^\star \text{ and } \mathbf{C} = \mathbf{C}^\star. \qquad [6.3]$$

Indeed,

$$\frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial S}{\partial \boldsymbol{X}}\right)^{\star}=\left\{\frac{\partial^2 S}{\partial x_i \partial x_j}\right\}$$

because $\left(\dfrac{\partial S}{\partial \boldsymbol{X}}\right)^{\star}=\begin{bmatrix}\dfrac{\partial S}{\partial x_1}\\ \vdots \\ \dfrac{\partial S}{\partial x_{n+1}}\end{bmatrix}$ and $\dfrac{\partial^2 S}{\partial x_i \partial x_j}=\dfrac{\partial^2 S}{\partial x_j \partial x_i}$.

Therefore, $\mathbf{A}$ is a symmetric matrix. This is the same for $\mathbf{C}$. Furthermore,

$$\mathbf{B}=\frac{\partial}{\partial \tilde{\boldsymbol{X}}}\left(\frac{\partial S}{\partial \boldsymbol{X}}\right)^{\star}=\left\{\frac{\partial^2 S}{\partial \tilde{x}_i \partial x_j}\right\},$$

Because $\det\left\{\dfrac{\partial^2 S}{\partial \tilde{x}_i \partial x_j}\right\}\neq 0$, where i is the column index and j is the row index, the matrix $\mathbf{B}$ is invertible. Furthermore, $\dfrac{\partial}{\partial \boldsymbol{X}}\left(\dfrac{\partial S}{\partial \tilde{\boldsymbol{X}}}\right)^{\star}=\left\{\dfrac{\partial^2 S}{\partial x_i \partial \tilde{x}_j}\right\}$, where i is the row index and j is the column index, and

$$\mathbf{B}=\frac{\partial}{\partial \tilde{\boldsymbol{X}}}\left(\frac{\partial S}{\partial \boldsymbol{X}}\right)^{\star}\qquad\Longrightarrow\qquad \mathbf{B}^{\star}=\frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial S}{\partial \tilde{\boldsymbol{X}}}\right)^{\star}.$$

From system [6.3],

$$\begin{cases} d\tilde{\boldsymbol{X}}=-\mathbf{B}^{-1}\mathbf{A}\,d\boldsymbol{X}+\mathbf{B}^{-1}d\boldsymbol{Y}\\ d\tilde{\boldsymbol{Y}}=\left(\mathbf{C}\mathbf{B}^{-1}\mathbf{A}-\mathbf{B}^{\star}\right)d\boldsymbol{X}-\mathbf{C}\mathbf{B}^{-1}d\boldsymbol{Y}\end{cases}$$

and

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}=\begin{bmatrix}-\mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1}\\ \mathbf{C}\mathbf{B}^{-1}\mathbf{A}-\mathbf{B}^{\star} & -\mathbf{C}\mathbf{B}^{-1}\end{bmatrix},$$

hence

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\,\dot{\mathbf{I}}\left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \\ \mathbf{C}\mathbf{B}^{-1}\mathbf{A}-\mathbf{B}^{\star} & -\mathbf{C}\mathbf{B}^{-1} \end{bmatrix} \times \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{A}\mathbf{B}^{\star-1} & \mathbf{A}\mathbf{B}^{\star-1}\mathbf{C}-\mathbf{B} \\ \mathbf{B}^{\star-1} & -\mathbf{B}^{\star-1}\mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}^{-1}\mathbf{B} \\ -\mathbf{B}^{-1}\mathbf{B} & \mathbf{0} \end{bmatrix} = \dot{\mathbf{I}}.$$

Consequently, the change of variables associated with a generating function S is canonical. ■

REMARK 6.7.– If the change of variables

$$\boldsymbol{Z} \in \mathcal{E}^{2n+2} \to \tilde{\boldsymbol{Z}} = \psi\left(\boldsymbol{Z}\right) \in \mathcal{E}^{2n+2}$$

is canonical, the change of variables

$$\tilde{\boldsymbol{Z}} \in \mathcal{E}^{2n+2} \;\to\; \boldsymbol{Z} = \psi^{-1}\left(\tilde{\boldsymbol{Z}}\right) \in \mathcal{E}^{2n+2}$$

is also canonical. Indeed,

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\,\dot{\mathbf{I}}\left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} = \dot{\mathbf{I}} \qquad \Longleftrightarrow \qquad \frac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}}\,\dot{\mathbf{I}}\left(\frac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}}\right)^{\star} = \dot{\mathbf{I}}.$$

EXAMPLE 6.4.– Consider the Jacobi transformation $S = a\,x^2 cotg\,\tilde{x}$ of two variables x and $\tilde{x}$. Let us verify that S is associated with a canonical change of variables, i.e. we obtain y and $\tilde{y}$ such that

$$dS = y\,dx - \tilde{y}\,d\tilde{x}.$$

We should have

$$\begin{cases} y = \dfrac{\partial S}{\partial x} = 2ax\,\mathrm{cotg}\,\tilde{x} \\ \tilde{y} = -\dfrac{\partial S}{\partial \tilde{x}} = \dfrac{ax^2}{\sin^2\tilde{x}} \end{cases} \qquad \Longrightarrow \qquad x^2 = \frac{\tilde{y}}{a}\sin^2\tilde{x} \;\text{ or }\; \begin{cases} x = \varepsilon\sqrt{\dfrac{\tilde{y}}{a}}\,\sin\,\tilde{x} \\ y = \varepsilon 2a\sqrt{\dfrac{\tilde{y}}{a}}\,\cos\,\tilde{x} \end{cases}$$

with $\varepsilon = \pm 1$. Indeed,

$$[x,y]_{(\tilde{x},\tilde{y})} = \frac{\partial x}{\partial \tilde{x}}\frac{\partial y}{\partial \tilde{y}} - \frac{\partial x}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial \tilde{x}} = \sqrt{\frac{\tilde{y}}{a}}\,\cos\,\tilde{x}\,\frac{2a}{\sqrt{a}}\frac{\cos\,\tilde{x}}{2\sqrt{\tilde{y}}}$$
$$-\frac{\sin\,\tilde{x}}{2\sqrt{a}\sqrt{\tilde{y}}}\left(-2a\sqrt{\frac{\tilde{y}}{a}}\,\sin\,\tilde{x}\right) = 1.$$

6.5. Remark on the symplectic scalar product

The scalar $\sigma\,(d\boldsymbol{Z})\,(\delta\boldsymbol{Z}) = d\boldsymbol{Z}^\star\,\dot{\mathbf{I}}\,\delta\boldsymbol{Z}$ is called the symplectic scalar product of vectors $d\boldsymbol{Z}$ and $\delta\boldsymbol{Z}$. If we write

$$d\boldsymbol{Z} = \begin{bmatrix} d\boldsymbol{X} \\ d\boldsymbol{Y} \end{bmatrix} \text{ and } \delta\boldsymbol{Z} = \begin{bmatrix} \delta\boldsymbol{X} \\ \delta\boldsymbol{Y} \end{bmatrix},$$

we obtain

$$\sigma\,(d\boldsymbol{Z})\,(\delta\boldsymbol{Z}) = [d\boldsymbol{X}^\star,\, d\boldsymbol{Y}^\star]\;\dot{\mathbf{I}}\;\begin{bmatrix} \delta\boldsymbol{X} \\ \delta\boldsymbol{Y} \end{bmatrix}$$
$$= [d\boldsymbol{X}^\star,\, d\boldsymbol{Y}^\star]\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{X} \\ \delta\boldsymbol{Y} \end{bmatrix} = [-d\boldsymbol{Y}^\star,\, d\boldsymbol{X}^\star]\begin{bmatrix} \delta\boldsymbol{X} \\ \delta\boldsymbol{Y} \end{bmatrix},$$

or

$$\sigma\,(d\boldsymbol{Z})\,(\delta\boldsymbol{Z}) = d\boldsymbol{X}^\star\delta\boldsymbol{Y} - d\boldsymbol{Y}^\star\delta\boldsymbol{X} = \delta\boldsymbol{Y}^\star d\boldsymbol{X} - \delta\boldsymbol{X}^\star d\boldsymbol{Y}.$$

This scalar product is the origin of the *symplectic geometry*.

THEOREM 6.5.– The symplectic scalar product is invariant under a canonical change of variables.

PROOF.– Term $\boldsymbol{Z} \in \mathcal{E}^{2n+2} \to \tilde{\boldsymbol{Z}} = \psi\,(\boldsymbol{Z}) \in \mathcal{E}^{2n+2}$ being a canonical change of variables,

$$\sigma\left(d\tilde{\boldsymbol{Z}}\right)\left(\delta\tilde{\boldsymbol{Z}}\right) = d\boldsymbol{Z}^\star\left[\left(\frac{\partial\tilde{\boldsymbol{Z}}}{\partial\boldsymbol{Z}}\right)^\star\dot{\mathbf{I}}\,\frac{\partial\tilde{\boldsymbol{Z}}}{\partial\boldsymbol{Z}}\right]\delta\boldsymbol{Z} = d\boldsymbol{Z}^\star\mathbf{L}\,\delta\boldsymbol{Z}$$

$$\text{where}\quad \mathbf{L} = \left(\frac{\partial\tilde{\boldsymbol{Z}}}{\partial\boldsymbol{Z}}\right)^\star\dot{\mathbf{I}}\,\frac{\partial\tilde{\boldsymbol{Z}}}{\partial\boldsymbol{Z}}.$$

Thus, since $\dot{\mathbf{I}}^2 = -\mathbf{1}$,

$$\mathbf{L}^{-1} = \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{-1} \dot{\mathbf{I}}^{-1} \left(\left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star}\right)^{-1} = \frac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}} \left(-\dot{\mathbf{I}}\right) \left(\frac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}}\right)^{\star}$$

Since the change of variables is canonical, $\dfrac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}}\, \dot{\mathbf{I}} \left(\dfrac{\partial \boldsymbol{Z}}{\partial \tilde{\boldsymbol{Z}}}\right)^{\star} = \dot{\mathbf{I}}$, therefore $\mathbf{L}^{-1} = -\dot{\mathbf{I}}$ and $\mathbf{L} = \dot{\mathbf{I}}$. Consequently,

$$\sigma\left(d\tilde{\boldsymbol{Z}}\right)\left(\delta\tilde{\boldsymbol{Z}}\right) = d\boldsymbol{Z}^{\star}\dot{\mathbf{I}}\,\delta\boldsymbol{Z} = \sigma\left(d\boldsymbol{Z}\right)\left(\delta\boldsymbol{Z}\right). \qquad \blacksquare$$

7

Properties of Phase Space

We consider the case where the Hamiltonian does not explicitly depend on time,

$$H = H\left(\boldsymbol{Q}, \boldsymbol{P}\right).$$

We call such a system the *Lagrangian dynamical system*.

7.1. Flow of a dynamical system

The phase space of a dynamical system was defined as the $2n$-dimensional space of coordinates $(\boldsymbol{Q}, \boldsymbol{P}) \equiv (q_1, \cdots, q_n, p_1, \cdots, p_n)$. In the case where $n = 1$, the space is a *phase plane*. The right-hand sides of Hamilton equations define a vector field: each point $(\boldsymbol{Q}, \boldsymbol{P})$ of the phase space is associated with a $2n$-component vector

$$\left(-\frac{\partial H}{\partial q_1}, \cdots, -\frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_n}\right)$$

and we assume that solutions of the Hamilton equations

$$\begin{cases} \dot{\boldsymbol{Q}} = \dfrac{\partial H}{\partial \boldsymbol{P}} \\ \dot{\boldsymbol{P}} = -\dfrac{\partial H}{\partial \boldsymbol{Q}} \end{cases} \qquad [7.1]$$

are defined over the time axis.

DEFINITION 7.1.– The flow associated with system [7.1] of the Hamilton equations is the mapping

$$g_t : (\boldsymbol{Q}(0), \boldsymbol{P}(0)) \longrightarrow (\boldsymbol{Q}(t), \boldsymbol{P}(t)),$$

where $\boldsymbol{Q}(t)$ and $\boldsymbol{P}(t)$ are solutions of the Hamilton equations that take the values $\boldsymbol{Q}(0)$ and $\boldsymbol{P}(0)$, respectively, at instant $t = 0$.

We have seen that g_t mappings constitute a Lie group.

EXAMPLE 7.1.– In the phase plane, the equation of motion is written as

$$\ddot{x} = f(x). \qquad [7.2]$$

The kinetic energy and the potential energy are

$$T = \frac{1}{2}\dot{x}^2 \quad \text{and} \quad W = -\int_{x_0}^{x} f(\zeta)\, d\zeta.$$

We denote $y = \dot{x}$ the conjugate variable of x. A solution of differential equation [7.2] is a solution of the system

$$\begin{cases} \dot{y} = f(x) \\ \dot{x} = y. \end{cases} \qquad [7.3]$$

The system of motion equations gives a vector field of the phase plane, which is called the *phase velocity vector field*. Similarly, in the general case of system [7.1], the right-hand side of the equations represents a vector field of $\mathbb{R}^{2n}$ also called the phase velocity vector field. A solution of system [7.3] in the form

$$t \in \mathbb{R} \longrightarrow \Psi(t) \in \mathbb{R}^2$$

is a representation of the motion in the phase space. At time t, the velocity is the phase velocity vector. The image obtained by mapping Ψ is the phase curve defined by parametric representation

$$\begin{cases} x = \psi(t) \\ y = \dot{\psi}(t). \end{cases}$$

Example: *Harmonic oscillator*.

The motions of a one-dimensional harmonic oscillator are solutions of the adimensional second-order differential equation

$$\ddot{x} + x = 0, \quad \text{associated with Hamilton's equations } \begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

where x denotes the position of the oscillator and

$$T = \frac{1}{2}\dot{x}^2, \quad W = \frac{1}{2}x^2 \quad \text{with} \quad \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 = E \quad \text{or} \quad x^2 + y^2 = 2E.$$

The trajectories in the phase plane are sets of constant energy E. They are concentric circles centered at the origin of the coordinate system (see Figure 7.1). At point $M\,(x, y)$ in the phase plane, the components of the phase velocity vector are $(-y, x)$. This vector is perpendicular to the radius vector of the circle. The motion in the phase plane is a uniform rotation around O where any set with constant energy represents a phase curve in the form

$$\begin{cases} x = r_0 \cos(t_0 - t) \\ y = r_0 \sin(t_0 - t)\,. \end{cases}$$

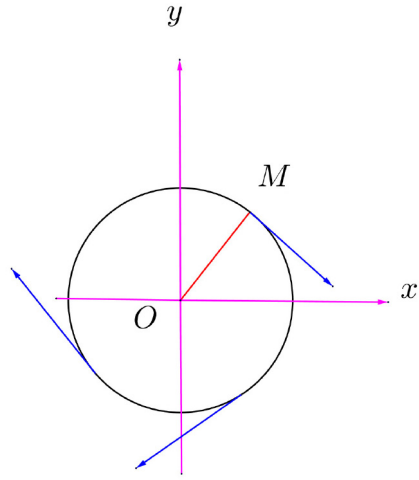


Figure 7.1. *Circle representing a trajectory of a harmonic oscillator in a phase plane. Phase velocity vectors are tangent to the trajectory. For a color version of the figures in this chapter, see www.iste.co.uk/gouin/mechanics.zip*

7.2. The Liouville theorem

7.2.1. *Preliminary*

LEMMA 7.1.– Given a matrix $\mathcal{A} = \{a_{ij}\}, i, j \in \{1, \cdots, n\}$ and the mapping

$$t \in R \longrightarrow \det(\mathbf{1} + \mathcal{A}\, t) \in \mathbb{R},$$

where $\mathbf{1}$ is the identity of $\mathbb{R}^n$; then,

$$\det(\mathbf{1} + \mathcal{A}\, t) = 1 + \mathrm{Tr}(\mathcal{A})\ t + t\,\varepsilon(t) \ \text{ where } \lim_{t\to 0} \varepsilon(t) = 0$$

and $\mathrm{Tr}(\mathcal{A}) = \sum_{i=1}^{n} a_{ii}$ denotes the trace of $\mathcal{A}$.

Indeed, $\det(\mathbf{1} + \mathcal{A}\, t)$ is an n-degree polynomial of t at the most, whose constant term is 1 and where the coefficient of t is $\sum_{i=1}^{n} a_{ii}$.

Consequences

Consider the first-order differential equation

$$\ddot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \qquad [7.4]$$

where $\boldsymbol{x} = (x_1, \cdots, x_n)$ is an n-component term of $\mathbb{R}^n$. Equation [7.4] is called autonomous; $\boldsymbol{f} = (f_1, \cdots, f_n)$ denotes the n functions from $\mathbb{R}^n$ to $\mathbb{R}$ assumed to be of class C^2. Equation [7.4] is equivalent to the system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, \cdots, x_n) \\ \qquad\qquad \vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, \cdots, x_n) \end{cases}$$

Then, $\boldsymbol{x} \in \mathbb{R}^n \longrightarrow \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ is the infinitesimal generator of the Lie group $\{\mathcal{G}_t, t \in R\}$, where t is the parameter of the differential equation [7.4]. The Lie group makes it possible to define the trajectory of [7.4]. Indeed, thanks to the definition of the infinitesimal generator,

$$\lim_{t\to 0} \frac{\mathcal{G}_t(\boldsymbol{x}) - \boldsymbol{x}}{t} = \boldsymbol{f}(\boldsymbol{x}),$$

or

$$\mathcal{G}_t(\boldsymbol{x}) = \boldsymbol{x} + t\,\boldsymbol{f}(\boldsymbol{x}) + t\,\boldsymbol{\alpha}(\boldsymbol{x},t) \text{ with for given } \boldsymbol{x},\ \lim_{t\to 0}\boldsymbol{\alpha}(\boldsymbol{x},t) = \boldsymbol{0}.$$

As the mapping $\boldsymbol{f}$ is assumed to be twice-continuously differentiable, theorems relative to systems of differential equations enable us to write

$$\mathcal{G}_t(\boldsymbol{x}) = \boldsymbol{x} + t\,\boldsymbol{f}(\boldsymbol{x}) + t^2\boldsymbol{\varepsilon}(\boldsymbol{x},t) \qquad \text{where } \lim_{t\to 0}\boldsymbol{\varepsilon}(\boldsymbol{x},t) = \boldsymbol{0},$$

and for given t and for any compact Δ of $\mathbb{R}^n$, $\boldsymbol{\varepsilon}(\boldsymbol{x},t)$ and $\dfrac{\partial\boldsymbol{\varepsilon}(\boldsymbol{x},t)}{\partial\boldsymbol{x}}$ are the bounded vector and linear mapping, respectively.

For any value of t, the associated group $\mathcal{G}_t$ defines the transformation of the space of parameters.

$$\boldsymbol{x} = (x_1, \cdots, x_n) \longrightarrow \mathcal{G}_t(\boldsymbol{x}).$$

Consider a measurable domain $\mathcal{D}_0$ in the space of parameters. Its image by the mapping $\mathcal{G}_t$ is denoted by $\mathcal{D}_t \equiv \mathcal{G}_t(\mathcal{D}_0)$. Let $\mathcal{V}_0$ be the volume of $\mathcal{D}_0$ and $\mathcal{V}_t$ be the volume of $\mathcal{D}_t$,

$$\mathcal{V}_0 = \int_{\mathcal{D}_0} dx \quad \text{and} \quad \mathcal{V}_t = \int_{\mathcal{D}_t} dx.$$

The determinant defines the measure dx of the volume. Variable changes in multiple integrals imply

$$\mathcal{V}_t = \int_{\mathcal{D}_0} \left|\det\left(\frac{\partial\mathcal{G}_t(\boldsymbol{x})}{\partial\boldsymbol{x}}\right)\right| dx.$$

Assuming that $\boldsymbol{x}$ belongs to a compact of $\mathbb{R}^n$,

$$\frac{\partial\mathcal{G}_t(\boldsymbol{x})}{\partial\boldsymbol{x}} = \boldsymbol{1} + t\,\frac{\partial\boldsymbol{f}(\boldsymbol{x})}{\partial\boldsymbol{x}} + t^2\frac{\partial\boldsymbol{\varepsilon}(\boldsymbol{x})}{\partial\boldsymbol{x}}(\boldsymbol{x},t)$$

and from Lemma 7.1, we deduce

$$\det\left(\frac{\partial\mathcal{G}_t(\boldsymbol{x})}{\partial\boldsymbol{x}}\right) = 1 + t\,\mathrm{Tr}\left(\frac{\partial\boldsymbol{f}(\boldsymbol{x})}{\partial\boldsymbol{x}}\right) + t^2\varepsilon_1(\boldsymbol{x},t) \text{ where } \quad \lim_{t\to 0}\varepsilon_1(\boldsymbol{x},t) = 0.$$

For an infinitesimal number t, we get

$$\det\left(\frac{\partial\mathcal{G}_t(\boldsymbol{x})}{\partial\boldsymbol{x}}\right) > 0$$

and

$$\mathcal{V}_t = \int_{\mathcal{D}_0} \left(1 + t\,\mathrm{Tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) + t^2\varepsilon_1(\boldsymbol{x}, t)\right) dx.$$

The volume $\mathcal{V}_t$ is derivable for $t = 0$, and its derivative value is

$$\mathcal{V}'(0) = \int_{\mathcal{D}_0} \mathrm{Tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) dx.$$

Similarly, we have

$$\mathcal{V}'(t)_{t=t_0} = \int_{\mathcal{D}_{t_0}} \mathrm{Tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) dx.$$

Indeed, the group $\{\mathcal{G}_t, t \in \mathbb{R}\}$ is additive, and we can replace 0 by $t - t_0$ with

$$\mathcal{G}_{t-t_0} \circ G_{t_0} = \mathcal{G}_t.$$

Furthermore,

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(x_1, \cdots, x_n) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x_1, \cdots, x_n) \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(x_1, \cdots, x_n) & \cdots & \dfrac{\partial f_n}{\partial x_n}(x_1, \cdots, x_n) \end{bmatrix}$$

and the divergence of the vector field $\boldsymbol{x} \in \mathbb{R}^n \longrightarrow \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ is

$$\mathrm{Tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x_1, \cdots, x_n).$$

Consequently,

$$\mathrm{Tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right) = \mathrm{div}\,\boldsymbol{f}(\boldsymbol{x}) \quad \text{and} \quad \mathcal{V}'(\mathrm{t})_{\mathrm{t=t_0}} = \int_{\mathcal{D}_{t_0}} \mathrm{div}\,\boldsymbol{f}(\boldsymbol{x})\,\mathrm{dx}.$$

THEOREM 7.1.– The transformation $\mathcal{G}_t$ conserves volume in the phase space if and only if $\mathrm{div}\,\boldsymbol{f}(\boldsymbol{x}) = 0$.

In a two-dimensional space and $\mathrm{div}\,\boldsymbol{f}(\boldsymbol{x}) = 0$, we can schematize the deformation from $\mathcal{D}_0$ to $\mathcal{D}_t$ by the domain occupied by *Arnold's cat*. The volume $\mathcal{V}_0$ of the cat at instant t_0 and the volume $\mathcal{V}_t$ at instant t are equal (see Figure 7.2).

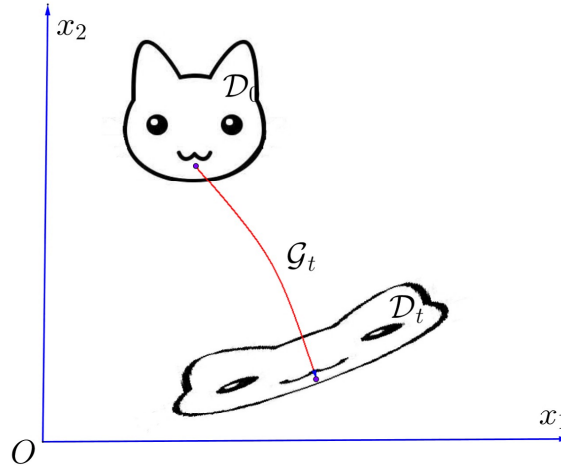


Figure 7.2. *Deformation of Arnold's cat in the phase plane using the mapping $\mathcal{G}_t$, where $\mathcal{D}_0$ denotes the initial position and $\mathcal{D}_t$ denotes the position at the instant t*

THEOREM 7.2.– ***The Liouville theorem.*** The flow $\mathcal{G}_t$ with $\operatorname{div} \boldsymbol{f}(\boldsymbol{x}) = 0$ conserves volume in the phase space. We can write

$$vol\ [\mathcal{G}_t\ (\mathcal{D}_0)] = vol\ \mathcal{D}_t.$$

7.2.2. *Application to mechanical systems*

The Hamiltonian is independent of time, and the space of $2n$ parameters constitutes the phase space. The equations of motions are given by Hamilton equations [7.1]. The infinitesimal operator of the differential system is

$$f(\boldsymbol{Q}, \boldsymbol{P}) = \begin{bmatrix} \dfrac{\partial H}{\partial \boldsymbol{P}} \\ \\ -\dfrac{\partial H}{\partial \boldsymbol{Q}} \end{bmatrix}.$$

Assuming that H is of class C^2,

$$\operatorname{div} \boldsymbol{f}(\boldsymbol{Q}, \boldsymbol{P}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\frac{\partial}{\partial \mathrm{q_i}} \left(\frac{\partial \mathrm{H}}{\partial \mathrm{p_i}} \right) - \frac{\partial}{\partial \mathrm{p_i}} \left(\frac{\partial \mathrm{H}}{\partial \mathrm{q_i}} \right) \right) = 0.$$

REMARK 7.1.– In the case where H depends on time, there is no longer any conservation of volume in the phase space. Any variation of volume can be interpreted as an energy dissipation.

EXAMPLE 7.2.– Examples of Liouville's theorem application.

Example 7.2a: For the oscillations of a harmonic oscillator, we have

$$\boldsymbol{f}(x,y) = \begin{bmatrix} y \\ -x \end{bmatrix} \text{ and } \operatorname{div}\boldsymbol{f}(\mathrm{x},\mathrm{y}) = 0.$$

The mapping $\mathcal{G}_t$ is a displacement in the phase plane, which conserves the volume. The domain $\mathcal{D}_t$ is represented in Figure 7.3.

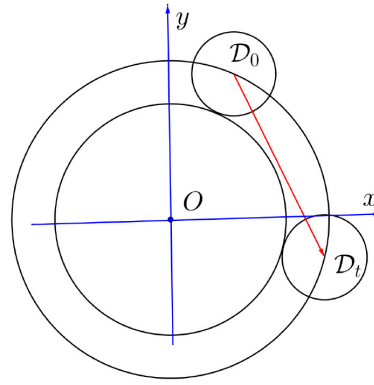


Figure 7.3. *Rotation from $\mathcal{D}_0$ to $\mathcal{D}_t$. The shape of domain $\mathcal{D}_t$ is unchanged by transformation $\mathcal{G}_t$*

Example 7.2b: The equation for an adimensional simple pendulum is

$$\ddot{x} + \sin x = 0,$$

which implies the energy equation

$$\frac{1}{2}\dot{x}^2 = \cos\, x + E.$$

Let us write $y^2 = 2\cos\, x + 2E$, where $y = \dot{x}$. The curves take the form represented in Figure 7.4. The bubble is transformed into the lunula of the same area.

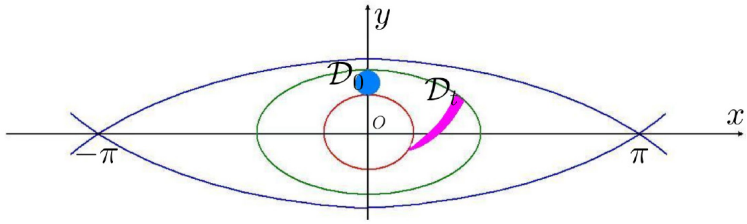


Figure 7.4. *Deformation from $\mathcal{D}_0$ to $\mathcal{D}_t$ by the mapping $\mathcal{G}_t$. The domain $\mathcal{D}_t$ is strongly deformed by deformation $\mathcal{G}_t$, but its area remains constant*

Example 7.2c: The equation of the adimensional inverted pendulum is

$$\ddot{x} = x \implies \begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}.$$

We deduce $y^2 - x^2 = 2E$. The trajectories are equilateral hyperbolas

$$\begin{cases} x = \lambda\, e^t + \mu\, e^{-t} \\ y = \lambda\, e^t - \mu\, e^{-t} \end{cases}. \qquad [7.5]$$

For $t = 0$, the initial conditions are $\begin{cases} x_0 = \lambda + \mu \\ y_0 = \lambda - \mu \end{cases}$, and finally,

$$\begin{cases} x = \dfrac{x_0 + y_0}{2}\, e^t + \dfrac{x_0 - y_0}{2}\, e^{-t} \\ y = \dfrac{x_0 + y_0}{2}\, e^t - \dfrac{x_0 - y_0}{2}\, e^{-t} \end{cases}.$$

We can determine the image of the domain $x^2 + (y-1)^2 \leq 1$ and can verify that its area is conserved by flow [7.5].

Example 7.2d: Consider the differential system

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

The system verifies $\operatorname{div} \boldsymbol{V} = 0$, where $\boldsymbol{V} = \begin{bmatrix} x \\ -y \end{bmatrix}$. We obtain the equilateral hyperbolas

$$\begin{cases} x = x_0\, e^t \\ y = y_0\, e^{-t} \end{cases}, \text{ consequently, } \quad x\, y = x_0\, y_0. \qquad [7.6]$$

In axes Ox and Oy, the images of disk $x_0^2 + y_0^2 \leq 1$ by [7.6] are elliptical domains with equation $x^2 e^{-2t} + y^2 e^{2t} \leq 1$. These domains have the same area as $x_0^2 + y_0^2 \leq 1$.

THEOREM 7.3.– In the phase space, a canonical change of variables keeps volumes unchanged.

We have seen that a canonical change of variables verifies

$$\boldsymbol{Z} \in \mathcal{E}^{2n+2} \rightarrow \tilde{\boldsymbol{Z}} = \psi\left(\boldsymbol{Z}\right) \in \mathcal{E}^{2n+2}$$

and satisfies the property

$$\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\, \dot{\mathbf{I}} \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} = \dot{\mathbf{I}}.$$

We deduce

$$\det \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}} \det\, \dot{\mathbf{I}} \det \left(\frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{\star} = \det\, \dot{\mathbf{I}} \quad \Longrightarrow \quad \left(\det \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right)^{2} = 1 \Longrightarrow \left|\det \frac{\partial \tilde{\boldsymbol{Z}}}{\partial \boldsymbol{Z}}\right| = 1,$$

and the volume is conserved. This property shows that in mechanical phase space, the concept of volume is intrinsic. This result is the starting point of the statistical mechanics.

7.3. The Poincaré recurrence theorem

7.3.1. *The recurrence theorem*

THEOREM 7.4.– Consider an affine Euclidean space $\mathcal{E}^n$. Given a bijective mapping g of $\mathcal{E}^n$ that conserves a closed bounded domain $D = g(D) \subset \mathcal{E}^n$ and an open set $U \subset D$ with non-zero finite volume, there exists a point $\boldsymbol{x} \in U$ and a positive integer p such that $g^p(\boldsymbol{x}) \in U$.

In the theorem, it is enough to write $g(D) \subset D$. Point $\boldsymbol{x}$ returns in the domain U after the transformation g has been iterated n times.

PROOF.– Sets $U, g(U), \cdots, g^k(U), \cdots$ denote the successive transformations of U by g. If we assume that for all $i, j \in N$, $g^i(U) \cap g^j(U) = \emptyset$, then $vol(D)$ is infinite because $vol\left(g^k(U)\right) = vol(U)$ and for any k, $vol(D) \geq k\ vol(U)$. Thus, there exist k and l $(k > l)$, positive integers such that $g^k(U) \cap g^l(U) \neq \emptyset$. Since the mapping g is bijective, $g^{-l}\left(g^k(U) \cap g^l(U)\right) = g^{k-l}(U) \cap U \neq \emptyset$. Let us write $\boldsymbol{y} = g^{k-l}(\boldsymbol{x})$ an element of $g^{k-l}(U) \cap U$, where $\boldsymbol{x} \in U$, which proves the theorem with $p = k - l$. ■

EXAMPLE 7.3.– Let D be a circle of center O and radius r in the Euclidean plane and g be the rotation of center O and angle α.

If $\alpha = 2\,m\,\pi/n$, where m and n are two positive integers, then g^n is the identity and the Poincaré theorem is verified.

If α is incommensurable with π, the Poincaré theorem implies that for any open interval of length ℓ of the circle, there exist $\boldsymbol{x}_0$ and n_0 such that $d\left(g^{n_0}(\boldsymbol{x}_0) - \boldsymbol{x}_0\right) < \ell$, where d measures the distance between the two points along the circle arc.

The result does not depend on $\boldsymbol{x}_0$. Let us denote by R_{θ_0} the rotation such that $R_{\theta_0} = g^{n_0}$, then $d\left(R_{\theta_0}\left(\boldsymbol{x}_0\right) - \boldsymbol{x}_0\right) < \ell$. Any point $\boldsymbol{x}$ of the circle can be written $\boldsymbol{x} = R_\theta\left(\boldsymbol{x}_0\right)$, consequently, $d\left(R_\theta R_{\theta_0}\left(\boldsymbol{x}_0\right) - R_\theta\left(\boldsymbol{x}_0\right)\right) < \ell$. Hence, $d\left(R_{\theta_0}\left(\boldsymbol{x}\right) - \boldsymbol{x}\right) < \ell$, or $d\left(g^{n_0}(\boldsymbol{x}) - \boldsymbol{x}\right) < \ell$. Here, we have $g^{n_0}(\boldsymbol{x}) \neq \boldsymbol{x}$ since α is incommensurable with π. Consequently, for any point $\boldsymbol{x}$ of D, the set of points $\{g^n(\boldsymbol{x}),\, n \in N\}$ is dense on D.

7.3.2. *Case of mechanics*

Given the differential equation

$$\ddot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \text{ where } \boldsymbol{x} \in R^n \qquad [7.7]$$

where $\boldsymbol{f}$ is continuously differentiable, we consider a closed bounded domain $\mathcal{D}$ in the phase space. The one-parameter group associated with [7.7] is $\{\mathcal{G}_t, t \in \mathbb{R}\}$. At time t, the mapping $\mathcal{G}_t$ is uniformly lipschitzian with a first-order derivative $\dfrac{\partial \mathcal{G}_t}{\partial \boldsymbol{x}}$ of elements $\{a_{ij}(\boldsymbol{x}, t) \in \mathbb{R},\ i, j \in \mathbb{R}^n\}$. Then,

$$\left\|\mathcal{G}_t(\boldsymbol{x}') - \mathcal{G}_t(\boldsymbol{x})\right\| \leq M \left\|\boldsymbol{x}' - \boldsymbol{x}\right\|, \qquad [7.8]$$

with $M = \sup |a_{ij}(\boldsymbol{x}, t)|$, where $\|\ \|$ represents the Euclidean norm and $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to $\mathcal{D}$. Theorem 7.4 can be rewritten as the following.

THEOREM 7.5.– ***The Poincaré recurrence theorem***. Equation [7.7] representing a mechanical system whose phase variables are included in a closed bounded domain $\mathcal{D}$ of $\mathbb{R}^n$, each point of the phase space returns to its initial position neighborhood.

The meaning of the theorem: the initial instant is $t = 0$. Given $\boldsymbol{x} \in \mathcal{D}$ and $t_0 > 0$, for any ball $\mathcal{O}$ centered at $\boldsymbol{x}$ and a strictly positive radius ε, there exists t_1, $t_1 > t_0$ such that $\mathcal{G}_{t_1}(\boldsymbol{x}) \in \mathcal{O}$.

The proof is as follows: for any $\boldsymbol{x}' \in \mathcal{D}$

$$\left\|\mathcal{G}_{t_1}(\boldsymbol{x}) - \boldsymbol{x}\right\| \leq \left\|\mathcal{G}_{t_1}(\boldsymbol{x}) - \mathcal{G}_{t_1}(\boldsymbol{x}')\right\| + \left\|\mathcal{G}_{t_1}(\boldsymbol{x}') - \boldsymbol{x}'\right\| + \left\|\boldsymbol{x}' - \boldsymbol{x}\right\|.$$

Given $\boldsymbol{x}$ and ball $\mathcal{O}_1$ of center $\boldsymbol{x}$ and radius $r = \texttt{inf}\left(\frac{\varepsilon}{4}, \frac{\varepsilon}{2(M+1)}\right)$, there exists t_1 (corresponding to t_0 increased by integer p given by Theorem 7.4) and $\boldsymbol{x}' \in \mathcal{O}_1$ such that $\mathcal{G}_{t_1}(\boldsymbol{x}') \in \mathcal{O}_1$. Then,

$$\|\mathcal{G}_{t_1}(\boldsymbol{x}') - \boldsymbol{x}'\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad (M+1)\,\|\boldsymbol{x}' - \boldsymbol{x}\| \leq \frac{\varepsilon}{2} \quad \Longrightarrow \quad \|\mathcal{G}_{t_1}(\boldsymbol{x}) - \boldsymbol{x}\| \leq \varepsilon.$$

We similarly demonstrate that any point in the phase space returns an infinite number of times into the neighborhood of its initial position. This theorem is one of the few conclusions for motions given by an equation of the form

$$\ddot{\boldsymbol{x}} = -\frac{\partial W}{\partial \boldsymbol{x}}$$

when condition [7.8] is verified and the solutions belong to a bounded domain.

EXAMPLE 7.4.– Consider a point M with the mass m whose initial position $M(0)$ is taken inside an axisymmetric half-bowl of axis Oz. Axis Oz is the ascendant vertical. Its position at the instant t is $M(t)$. We have

$$T + W = E \quad \Longleftrightarrow \quad \frac{1}{2}\,m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = E.$$

In conjugate variables p, q, r,

$$\frac{1}{2m}\,(p^2 + q^2 + r^2) + mgz = E.$$

For $z \geq 0$, the triplet (p, q, r) is bounded. For an equation on the regular surface $f(x, y, z) = 0$ of the half-bowl, x, y and z are bounded. The material point, which is experimentally constituted by a small marble rolling without friction inside the half-bowl, comes infinitively back as closely as we wish near both the initial position and the velocity (see Figure 7.5).

REMARK 7.2.– A paradoxical consequence of the Poincaré recurrence theorem: consider a closed receptacle made up of two chambers separated by a partition wall. The first chamber is filled with gas. The second chamber is empty. The gas molecules are subject to laws of forces deriving from a potential. We open the partition wall separating the two chambers. After a time T, the gas molecules which have spread across both chambers will again congregate in the first chamber. The paradox is due to the fact that time T taken for all the gas molecules to return to the first chamber is greater than the existence time of the universe.

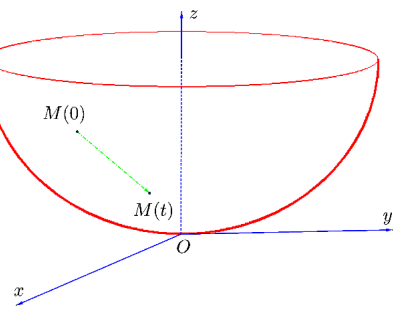


Figure 7.5. *Point M subject to gravity rolls without friction inside an axisymmetric half-bowl*

8

Oscillations and Small Motions of Mechanical Systems

8.1. Preliminary remarks

The Cauchy problem of differential equations does not always lead to easy problems. The fact can be demonstrated by using two simple examples.

EXAMPLE 8.1.– The differential equation

$$y' - y = 0$$

associated with the function

$$x \in \mathbb{R} \longrightarrow y(x) \in \mathbb{R}$$

has the general solution $y = \lambda\, e^x$ with $\lambda \in \mathbb{R}$.

Let us choose $y(0) = 0$ as initial condition for $x = 0$; thus, $\lambda = 0$, and the solution is $y(x) = 0$.

Let us now choose as initial condition $y(0) = \varepsilon$ for $x = 0$ with $0 < \varepsilon \ll 1$; then, the solution is $y(x) = \varepsilon\, e^x$.

Although the initial conditions differ very slightly, the distance between the two solutions goes to infinity when x goes to infinity.

EXAMPLE 8.2.– The differential equation

$$y'' + \omega^2 y = 0 \quad \text{with} \quad \omega \in \mathbb{R}^{+*}$$

associated with the function

$$x \in \mathbb{R} \longrightarrow y(x) \in \mathbb{R}$$

has the general solution $y = a\,\cos(\omega\,x) + b\,\sin(\omega\,x)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Let us choose $y(0) = 0$ and $y'(0) = 0$ as initial conditions for $x = 0$, the solution is $y(x) = 0$.

Let us now choose $y(0) = \varepsilon$, where $0 < \varepsilon \ll 1$ and $y'(0) = 0$ as initial conditions for $x = 0$; then, the solution is $y(x) = \varepsilon\,\cos(\omega\,x)$.

The distance between the two solutions is of ε-order.

Now, let us disturb the value of ω as $(1+\varepsilon)\,\omega$, with $0 < \varepsilon \ll 1$. Let us choose $y(0) = 1$ and $y'(0) = 0$ as initial conditions for $x = 0$; the solutions are $y(x) = \cos(\omega\,x)$ and $y(x) = \cos\big((1+\varepsilon)\,\omega\,x\big)$, respectively.

Note that $\cos\big((1+\varepsilon)\,\omega\,x\big) - \cos(\omega\,x)$ is not of ε-order. To prove this property, let us choose $\omega\,x = \dfrac{k\pi}{2}$, where $k \in 2N+1$ (k is any odd integer) and $\varepsilon\dfrac{k\pi}{2}$ close to 2π by choosing the nearest odd integer to $\dfrac{4}{\varepsilon}$. We obtain

$$\big|\cos\big((1+\varepsilon)\,\omega\,x\big) - \cos(\omega\,x)\big| > \frac{1}{2}.$$

The linear differential equations are the easiest to solve, so the theory of linear oscillations is an important part of mechanics. However, in many problems, the differential equations are not linear, but their linearization often provides a satisfactory approximated solution. While this is not true for all problems, the study of linearized equations is an important step in solving nonlinearized problems. It would be informative to consult the examples given in Arnold's book, "*Ordinary Differential Equations*" (Arnold 1992).

8.2. The Weierstrass discussion

8.2.1. *Introduction*

In many problems, the study of the motion of material systems leads to the differential equation

$$\ddot{q} = g(q), \qquad [8.1]$$

where q is a real function of the real variable t, and g is a real function of variable q which depends on the integration constants given by linear first integrals. In mechanics, this is a consequence of the case called the *regular integration case*. The discussion of the differential equation [8.1] is called the *Weierstrass discussion*. The discussion ends with the concept of the stability of equilibrium positions. Equation [8.1] implies

$$\ddot{q}\,\dot{q} = g(q)\,\dot{q}.$$

Function g is assumed continuous; we obtain

$$\dot{q}^2 = 2\int_{q_0}^{q} g(u)du + c, \qquad [8.2]$$

where the constant $c = \dot{q}_0^2$ is obtained at initial instant $t = t_0$ corresponding to the position $q = q_0$. We deduce the *fundamental equation*

$$\dot{q}^2 = f(q) \qquad [8.3]$$

where $f(q) = 2\displaystyle\int_{q_0}^{q} g(u)du + c$. The C^1-function f generally depends on the initial conditions and on the total energy yielded by the Painlevé theorem.

REMARK 8.1.– Equation [8.3] is equivalent to

$$\dot{q}\,(\ddot{q} - g(q)) = 0 \quad \Longleftrightarrow \quad \begin{cases} \dot{q} = 0, \\ \text{or} \\ \ddot{q} = \dfrac{1}{2} f'(q)\,. \end{cases}$$

Equation [8.3] introduces solutions

$$\dot{q} = 0 \quad \Longleftrightarrow \quad q\,(t) = q_0,$$

where q_0 is a constant. In order for the function $q\,(t) = q_0$ to be a solution of [8.1], it is necessary and sufficient that $f'\,(q_0) = 0$. Simultaneously, we must have

$$\begin{cases} f(q_0) = 0, \\ \\ f'(q_0) = 0, \end{cases} \qquad [8.4]$$

The motion $q\,(t) = q_0$, or the equilibrium position for the parameter q_0, must satisfy system [8.4]. The equilibrium positions of the parameter q_0 correspond to the multiple roots of $f(q) = 0$.

Nevertheless, in order for q_0 to be the equilibrium position, it is necessary and sufficient that $f'(q_0) = 0$. The additional relation $f(q_0) = 0$ gives the value of the integration constant of equation [8.2].

Let us also remark that if $p = \dot{q}$, we obtain the differential system

$$\begin{cases} \dot{q} = p, \\ \\ \dot{p} = \dfrac{1}{2}\, f'(q), \end{cases}$$

whose equilibrium positions correspond to $p = 0$ and $f'(q) = 0$, or equivalently $f(q) = 0$ and $f'(q) = 0$.

8.2.2. *Discussion of fundamental equation* [8.3]

The variations of q only occur in intervals where $f(q) \geq 0$. Then, equation [8.3] becomes

$$dt = \varepsilon \frac{dq}{\sqrt{f\,(q)}} \quad \text{where} \quad \varepsilon = \pm 1.$$

The choice of ε depends on the motion direction at initial instant t_0. Assume that at instant t_0, $\dot{q}_0 > 0$, then $\dot{q}_0 = \sqrt{f(q_0)}$, where $q(t_0) = q_0$ yields the value of c in equation [8.2]. The motion is given by

$$t - t_0 = \int_{q_0}^{q} \frac{du}{\sqrt{f(u)}}. \qquad [8.5]$$

The relation remains valid for all values of q greater than q_0 if $f\,(q)$ does not reach a zero of equation $f(u) = 0$.

So, two cases are considered.

First case: for any q greater than q_0, the function f is strictly positive. The motion verifies time law [8.5]. Then, along the motion, t is a strictly increasing function of q; conversely, q is a strictly increasing function of t. The position "$q = +\infty$" is reached in a finite or an infinite time depending on the nature of integral $\displaystyle\int_{q_0}^{+\infty} \frac{du}{\sqrt{f(u)}}$.

– If the integral is divergent, the position "$q = +\infty$" is reached in an infinite time and the study is ended. The motion is the strictly increasing mapping

$$t \in [t_0, +\infty[\quad \longrightarrow \quad q(t) \in [q_0, +\infty[.$$

– If the integral is convergent, the position "$q = +\infty$" is reached in a finite time t_1 such that

$$t_1 - t_0 = \int_{q_0}^{+\infty} \frac{du}{\sqrt{f(u)}}.$$

It is necessary to revisit the physical definition of the problem.

Second case: There exists q_1, first root of equation $f(u) = 0$ greater than q_0. Therefore, $\forall\, q \in [q_0, q_1[,\ f(q) > 0$. The nature of integral $\displaystyle\int_{q_0}^{q_1} \frac{du}{\sqrt{f(u)}}$ specifies if the position $q = q_1$ is reached in a finite or an infinite time.

Case *(a)*: q_1 is a multiple root of the equation $f(u) = 0$. The integral is divergent; the position $q = q_1$ is reached in an infinite time and the problem is ended. Let us note that the position $q = q_1$ corresponds to the definition of the equilibrium position of equation [8.3].

Case *(b)*: q_1 is a simple root of equation $f(u) = 0$. The integral is convergent; the position $q = q_1$ is reached at time

$$t_1 = t_0 + \int_{q_0}^{q_1} \frac{du}{\sqrt{f(u)}}.$$

However, the position $q = q_1$ is not a multiple root of the equation $f(u) = 0$ and consequently is not an equilibrium position of equation [8.3]. Since f is a continuous function, $f(q) < 0$, in a neighborhood of q_1 with $q > q_1$. Consequently, for $t > t_1$, the motion is represented by the equation

$$\dot{q} = -\sqrt{f\,(q)},$$

which implies

$$t = t_1 - \int_{q_1}^{q} \frac{du}{\sqrt{f(u)}}, \quad \text{where} \quad q < q_1.$$

For the position $q = q_1, \dot{q}_1 = 0$ and $\ddot{q}_1 \neq 0$; from the variation of $q(t)$ for $t > t_1$, $\ddot{q}_1 = \frac{1}{2} f'(q_1) < 0$. The study of the motion is obtained in a same way as previous cases; it depends on the possible root for $q < q_1$ of equation $f(q) = 0$. The parameter $q(t)$ takes the same positions at times smaller than t_1, but with velocities of opposite sign. The position q, where $q < q_1$, is reached at two instants $t_1 - \tau$ and $t_1 + \tau$ that are symmetric with respect to t_1,

$$\tau = \int_{q_1}^{q} \frac{du}{\sqrt{f(u)}}.$$

The position $q(t)$ is an even function of $t - t_1$. All cases may occur again for $t > t_1$:

– Case *(c)*: Uninterrupted motion until "$q = -\infty$".

– Case *(d)*: Motion goes up to the root q_2 of the equation $f(u) = 0$ (where $q_2 < q_0$).

When q_2 is a multiple root of equation $f(u) = 0$, the position $q = q_2$ is reached in an infinite time.

When q_2 is a simple root of equation $f(u) = 0$, the position $q = q_2$ is reached at the instant t_2 such that

$$t_2 = t_1 - \int_{q_1}^{q_2} \frac{du}{\sqrt{f(u)}} = t_1 + \int_{q_2}^{q_1} \frac{du}{\sqrt{f(u)}}.$$

This case corresponds to q_0 located between two simple roots of equation $f(u) = 0$. The motion is periodic. The period of an oscillation is the time required to return to the position q_0 with the initial velocity $\dot{q}_0$. The period $\mathcal{T}$ is

$$\mathcal{T} = \int_{q_0}^{q_1} \frac{du}{\sqrt{f(u)}} - \int_{q_1}^{q_2} \frac{du}{\sqrt{f(u)}} + \int_{q_2}^{q_0} \frac{du}{\sqrt{f(u)}} = 2 \int_{q_2}^{q_1} \frac{du}{\sqrt{f(u)}}.$$

The period $\mathcal{T}$ does not depend on q_0.

The case $\dot{q}_0 = 0$ does not present difficulty. It is enough to consider the sign of $f'(q_0)$ to deduce the direction of the variation of $q(t)$.

8.2.3. *Graphical interpretation*

We represent the analysis by using graphs.

a) $f(q) = 0$ has no root (for all $q,\ f(q) > 0$).

The asymptotic directions of $q(t)$ are obtained as a limit of $\sqrt{f\,(q)}$ when q goes to $+\infty$ or $-\infty$ (see Figure 8.1).

b) $f(q) = 0$ has a simple root q_1 (see Figure 8.1).

c) $f(q) = 0$ has a multiple root q_1 (see Figure 8.2).

d) $f(q) = 0$ has two simple roots q_1 and q_2. The motion is oscillatory of period $\mathcal{T} = 2\displaystyle\int_{q_2}^{q_1} \frac{du}{\sqrt{f(u)}}$ (see Figure 8.2).

e) $f(q) = 0$ has a simple root q_1 and a multiple root q_2 (see Figure 8.3).

f) $f(q) = 0$ has two multiple roots q_1 and q_2 (see Figure 8.3).

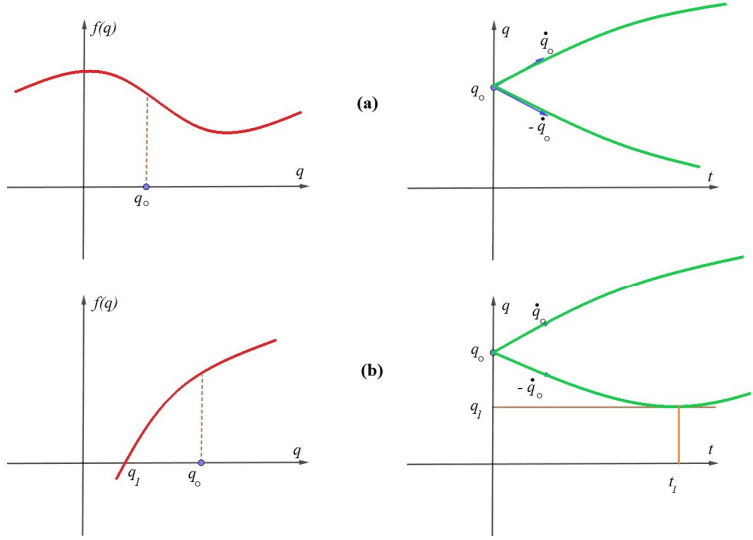


Figure 8.1. *Weierstrass discussion for cases a and b: The lines plotted on the left-hand side represent the function $q \to f(q)$ for q belonging to a domain where $f(q) > 0$. The lines plotted on the right-hand side correspond to the motions associated with the initial condition $q(t_o) = q_o$ and for the values of $\dot{q}_o = \pm\sqrt{f(q_o)}$. For a color version of the figures in this chapter, see www.iste.co.uk/gouin/mechanics.zip*

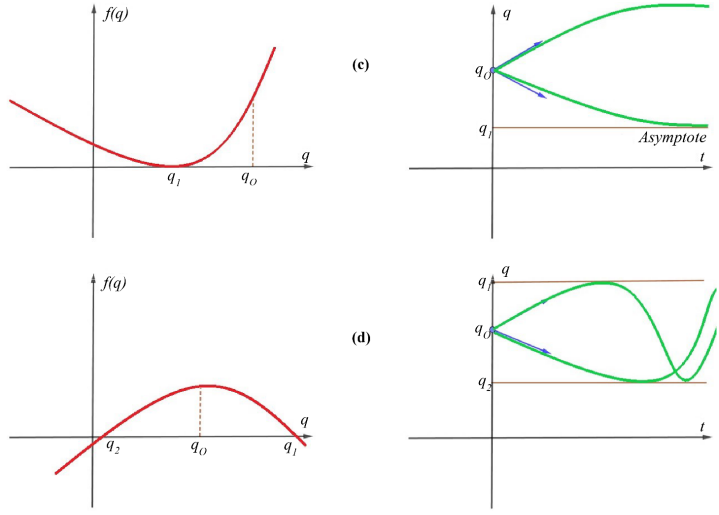


Figure 8.2. *The Weierstrass discussion for cases c and d. The description of the curves is the same as in Figure 8.1*

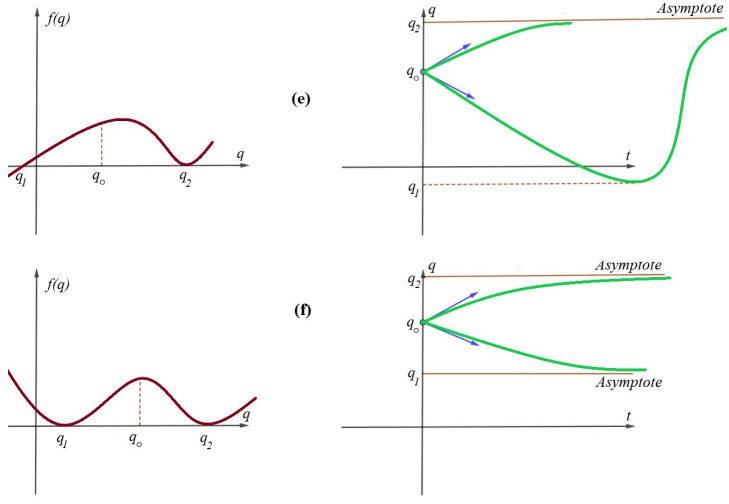


Figure 8.3. *Weierstrasss discussion in the cases e and f. The description of the curves is the same as in Figure 8.1*

8.2.4. *Study of equilibrium positions*

Assume that $q(t) = q_0$ is an equilibrium position. The equation associated with the equilibrium position satisfies

$$\dot{q}^2 = f(q) \text{ with } f(q_0) = 0 \text{ and } f'(q_0) = 0.$$

The second-order equation of motion

$$\ddot{q} = \frac{1}{2} f'(q) \qquad [8.6]$$

enables us to write the fundamental equation for any other motions in the form

$$\dot{q}^2 = f(q) - h,$$

where h is a constant depending on initial conditions and which is zero when $q(t) = q_0$. Constant h is a perturbation of equation [8.6] with respect to the equilibrium position. It corresponds to the idea that "small values" of h are associated with "neighbor" motions of $q(t) = q_0$.

DEFINITION 8.1.– The definition is only applicable to the case of equation [8.3]: the position q_0 is a stable equilibrium position if and only if

$$\forall \varepsilon \in \mathbb{R}^{+*}, \exists \eta \in \mathbb{R}^{+*}, \forall h \in \mathbb{R}, |h| \leq \eta \implies \forall t, t \geq t_0, \|q(t) - q_0\| \leq \varepsilon.$$

An equilibrium is said to be unstable if it is not stable.

Assuming that f is developed near q_0 to second order by the Taylor–Lagrange formula, and $F(q) = f(q) - h$, there exists $\theta \in]0, 1[$ such that

$$F(q) = F(q_0) + (q - q_0) F'(q_0) + \frac{(q - q_0)^2}{2} F''(q_0 + (q - q_0)\theta).$$

From $\quad F(q_0) = -h, \quad F'(q_0) = f'(q_0) = 0, \quad F''(q) = f''(q),$

we obtain $\quad F(q) = -h + \dfrac{(q - q_0)^2}{2} F''(q_0 + (q - q_0)\theta).$

Function f'' is continuous at q_0 and it is assumed that $f''(q_0) \neq 0$. In this case, the study of stability becomes a graphical study (see Figure 8.4). A more precise study in the *sense of Lyapunov* is done in later sections; however, this graphical representation is instructive as an analysis of stability.

– **First case**: $f''(q_0) < 0$ For $h > 0$, there is no physical motion. Since F is continuous, for $h < 0$, the amplitude of the motion is infinitesimal with respect to h; the equilibrium is stable.

– **Second case**: $f''(q_0) > 0$ For $h > 0$, the motion is associated with arcs AC and BD. For $h < 0$, we obtain the whole curve; the equilibrium is unstable.

– **Case** $f''(q_0) = 0$: The function f is assumed to be sufficiently regular, p is the first non-zero derivative at q_0. This derivative is assumed to be continuous in the neighborhood of q_0. The Taylor–Lagrange development to the pth order yields:

$$F(q) = -h + \frac{(q-q_0)^p}{p!} F^{(p)}\left(q_0 + (q-q_0)\,\theta\right).$$

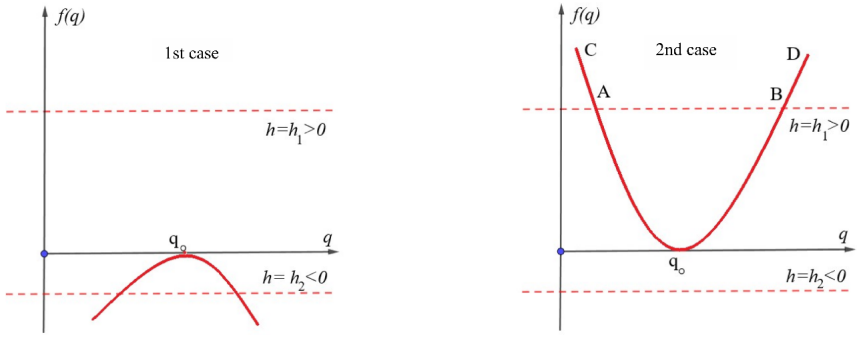


Figure 8.4. *Small motions around an equilibrium position* $q = q_0$ *following the sign of the concavity of curve* $f(q)$

The following graphical discussion can be deduced:

If p is even, we deduce a graphical discussion analogous to that in Figure 8.4. Only the contact order of the function f with the q axis is modified.

For $f^{(p)}(q_0) < 0$, the equilibrium is stable.

For $f^{(p)}(q_0) > 0$, the equilibrium is unstable.

If p is odd, the study near $q = q_0$ is shown on the graph in Figure 8.5; the equilibrium is unstable.

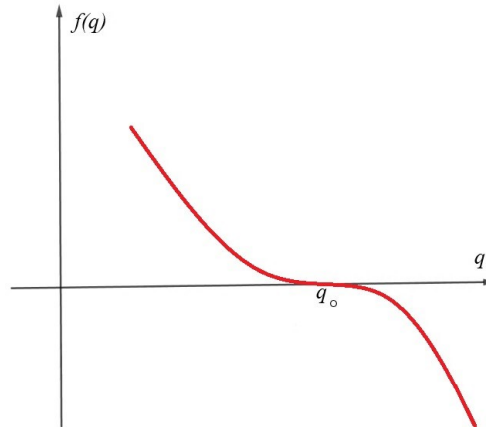


Figure 8.5. *Graph of $f(q)$ near $q = q_0$ when p is odd*

8.2.5. *Small motions near an equilibrium position*

We are in the simplest case of a stable equilibrium position:

$$f(q_0) = 0,\ f'(q_0) = 0,\ f''(q_0) < 0.$$

The general equation of motion associated with parameter q is [8.1], written in the form

$$\ddot{q} = \frac{1}{2} f'(q).$$

The finite-increments formula applied to f' implies

$$\exists\, \theta \in]0,1[\quad \text{such that} \quad f'(q) = f'(q_0) + (q - q_0)\, f''(q_0 + \theta\,(q - q_0))\,.$$

Therefore,

$$\ddot{q} - \frac{(q - q_0)}{2} f''(q_0 + \theta\,(q - q_0)) = 0.$$

We call a linearized equation of equation [8.1] near the equilibrium position $q = q_0$ the second-order differential equation with constant coefficients:

$$\ddot{q} - \frac{(q - q_0)}{2} f''(q_0) = 0$$

or

$$\ddot{u} + \omega_0^2\, u = 0 \qquad \text{with } u = q - q_0 \quad \text{and} \quad \omega_0^2 = -\frac{f''(q_0)}{2}. \tag{8.7}$$

The motion is periodic of period

$$\mathcal{T}_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{2}{-f''(q_0)}}, \tag{8.8}$$

$\mathcal{T}_0$ period of linearized equation [8.7] is called *period of small oscillations*.

REMARK 8.2.– It is possible to get the result from equation [8.3]:

$$\dot{u}^2 = f''(q_0 + \theta u)\,\frac{u^2}{2} - h.$$

The corresponding linearized equation is

$$\dot{u}^2 = f''(q_0)\,\frac{u^2}{2} - h.$$

According to the study in section 8.2.2, the period of oscillations is

$$2\int_{-u_1}^{+u_1} \sqrt{\frac{2}{-f''(q_0)}}\,\frac{du}{\sqrt{u_1^2 - u^2}} = 2\sqrt{\frac{2}{-f''(q_0)}}\left[\arcsin\frac{u}{u_1}\right]_{-u_1}^{+u_1},$$

where

$$u_1 = \sqrt{\frac{2h}{f''(q_0)}}$$

which allows us to get the period $\mathcal{T}_0$.

EXAMPLE 8.3.– ***The simple pendulum***. A point $\boldsymbol{M}$ with mass m moves without friction along a fixed, vertical hoop with radius R. Axes $\boldsymbol{O\,i\,j}$ are orthonormal and $\boldsymbol{Oi}$ denotes the descending vertical; we write $\boldsymbol{OM} = R\,\boldsymbol{u}$, $\boldsymbol{u}$ being a unit vector; let $\boldsymbol{p}$ be the unit vector directly perpendicular to $\boldsymbol{u}$ and θ be the oriented angle $\langle \boldsymbol{i}, \boldsymbol{u} \rangle$ (see Figure 8.6). Then, $\boldsymbol{v} = R\,\dot{\theta}\,\boldsymbol{p}$ is the velocity of $\boldsymbol{M}$ and $m\,\boldsymbol{a} = m\,R\,\ddot{\theta}\,\boldsymbol{p} - m\,R\,\dot{\theta}^2\boldsymbol{u}$ is the impulsion or acceleration quantity. The point $\boldsymbol{M}$ is submitted to its weight $mg\,\boldsymbol{i}$ and to the perfect reaction $N\,\boldsymbol{u}$ normal to the hoop. The equation of motion is

$$m\,g\,\boldsymbol{i} + N\boldsymbol{u} = m\,R\,\ddot{\theta}\,\boldsymbol{p}\; - m\,R\,\dot{\theta}^2\boldsymbol{u}.$$

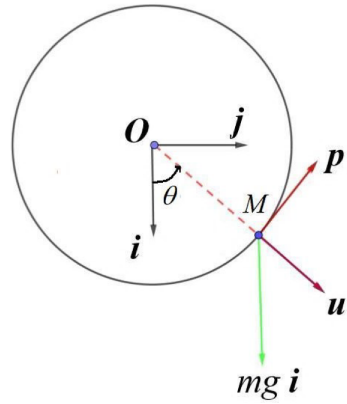


Figure 8.6. *Diagram of the simple pendulum. The point M moves without friction along a vertical circle whose center is O and radius is R*

By projecting on $\boldsymbol{p}$ and $\boldsymbol{u}$, we obtain, respectively

$$\begin{aligned} m\,R\,\ddot{\theta} + m\,g\sin\theta &= 0 \\ -m\,R\,\dot{\theta}^2 - m\,g\cos\theta &= N \end{aligned}$$

The first equation can be written

$$\ddot{\theta} + (g/R)\sin\theta = 0$$

whose linearized equation near the stable equilibrium position $\theta = 0$ is

$$\ddot{\theta} + (g/R)\,\theta = 0.$$

The small-oscillation period is $\mathcal{T} = 2\pi\sqrt{R/g}$. So, previous graphical discussion can be used.

REMARK 8.3.– In section 8.2.4, we assumed that solutions of the linearized equation are linearized solutions of the initial equation. This property is not always true. Indeed, the limited development of the derivative is not always the derivative of its limited development. In the case of the simple pendulum, it is possible to use mathematical tools to prove that the period of oscillations goes to the period of small oscillations when h goes to zero. The theory of small motions is related with the stability of dynamical systems. The problem is still partially open. For simple differential equations, the question does not always have an answer. We will study the question with more detail.

8.3. Equilibrium position of an autonomous differential equation

DEFINITION 8.2.– A differential equation is autonomous when written in the form of

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}), \qquad [8.9]$$

where $\boldsymbol{x}$ belongs to an open set $\mathcal{O}$ of $\mathbb{R}^n$ and $\boldsymbol{f}$ continuous at $\mathbf{0}$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Equation [8.9] represents a system of first-order differential equations of infinitesimal operator $\boldsymbol{f}(\boldsymbol{x})$.

DEFINITION 8.3.– The position $\boldsymbol{x}_0$ belonging to open set $\mathcal{O}$ is the equilibrium position of differential equation [8.9] if and only if, for all t

$$\boldsymbol{x}(t) = \boldsymbol{x}_0$$

satisfies differential equation [8.9], with $\boldsymbol{x}_0$ being a constant vector.

Thus, $\boldsymbol{x}(t) = \boldsymbol{x}_0$ is the equilibrium position if and only if $\boldsymbol{f}(\boldsymbol{x}_0) = 0$. In mechanics, the equilibrium positions correspond to positions whose phase velocity is zero.

Let us review the well-known cases.

Case of Lagrangian dynamical systems: Consider a mechanical system of n independent parameters $\boldsymbol{Q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$, where $\boldsymbol{Q}$ belongs to open set $\mathcal{O}$. The system admits a Lagrangian

$$L(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) = T(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) - W(\boldsymbol{Q})$$

where $T(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ and $W(\boldsymbol{Q})$ denote the kinetic and potential energies, respectively. Since

$$T(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) = \frac{1}{2}\, \dot{\boldsymbol{Q}}^{\star} \boldsymbol{\mathcal{A}}\, \dot{\boldsymbol{Q}} = \frac{1}{2} \sum_{i,j} a_{ij}(q_1, \cdots, q_n)\, \dot{q}_i\, \dot{q}_j$$

where $\mathcal{A} = \{a_{ij}\}$, with $i, j \in \{1, \cdots, n\}$, is an invertible symmetric matrix associated with a positive definite quadratic form differentiable on $\mathcal{O}$, and

$$W(\boldsymbol{Q}) = W(q_1, \cdots, q_n)$$

is also differentiable on $\mathcal{O}$. Let us recall that as $p_j = \dfrac{\partial T}{\partial \dot{q}_j}$ is the polar half-form of T, the conjugate variables of $q_1, \cdots, q_n$ are

$$p_j = \sum_{i=1}^{n} a_{ij}\dot{q}_i \quad \text{or} \quad \boldsymbol{P} = \mathcal{A}\,\dot{\boldsymbol{Q}}.$$

We obtain

$$T = \frac{1}{2}\,\boldsymbol{P}^{\star}\mathcal{A}^{-1}\,\boldsymbol{P}$$

where $\mathcal{A}^{-1}$ is a positive definite quadratic form with respect to $\boldsymbol{P}$. The equations of motion are the n Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{Q}}}\right) = \frac{\partial L}{\partial \boldsymbol{Q}}\,.$$

Since $\boldsymbol{P} = \left(\dfrac{\partial L}{\partial \dot{\boldsymbol{Q}}}\right)^{\star}$ and $\dfrac{\partial L}{\partial \boldsymbol{Q}} = \dfrac{\partial T}{\partial \boldsymbol{Q}} - \dfrac{\partial W}{\partial \boldsymbol{Q}}$, we obtain a system equivalent to the system of Hamilton equations,

$$\begin{cases} \dot{\boldsymbol{P}} = \left(\dfrac{\partial T}{\partial \boldsymbol{Q}}\right)^{\star} - \left(\dfrac{\partial W}{\partial \boldsymbol{Q}}\right)^{\star} \\ \dot{\boldsymbol{Q}} = \mathcal{A}^{-1}\,\boldsymbol{P} \end{cases} \qquad [8.10]$$

The equilibrium positions of system [8.10] satisfy $\mathcal{A}^{-1}\,\boldsymbol{P} = \boldsymbol{0}$; thus, $\boldsymbol{P} = \boldsymbol{0}$ and $\dfrac{\partial T}{\partial \boldsymbol{Q}} - \dfrac{\partial W}{\partial \boldsymbol{Q}} = \boldsymbol{0}$. Since $\dot{\boldsymbol{Q}} = \boldsymbol{\mathcal{A}^{-1}}\,\boldsymbol{P}$, $\boldsymbol{P} = 0$ is equivalent to $\dot{\boldsymbol{Q}} = \boldsymbol{0}$. It can be verified, by differentiating the quadratic form giving T as a function of $\boldsymbol{Q}$ and $\dot{\boldsymbol{Q}}$, that the associated linear form $\dfrac{\partial T(\boldsymbol{Q}, \dot{\boldsymbol{Q}})}{\partial \boldsymbol{Q}}$ is zero.

Therefore, the equilibrium equations satisfy

$$\boldsymbol{P} = \boldsymbol{0} \quad \text{and} \quad \frac{\partial W(\boldsymbol{Q})}{\partial \boldsymbol{Q}} = \boldsymbol{0}.$$

A critical point of potential energy verifies $\dfrac{\partial W(\boldsymbol{Q})}{\partial \boldsymbol{Q}} = \boldsymbol{0}$, and we obtain the theorem:

THEOREM 8.1.– Equilibrium positions of a Lagrangian dynamical system are given by $\{\boldsymbol{P}_0 = 0,\ \boldsymbol{Q}_0\}$, where $\boldsymbol{Q}_0$ is a critical point of potential energy W.

8.4. Stability of equilibrium positions of an autonomous differential equation

Our aim is to find motions whose initial conditions are close to those of an equilibrium position. Time t_0 is the initial instant.

DEFINITION 8.4.– ***The Lyapunov stability***. An equilibrium position $\boldsymbol{x}_0$ of system [8.9] is Lyapunov stable if and only if for any neighborhood $\mathcal{V}$ of $\boldsymbol{x}_0$, there exists a neighborhood $\mathcal{W}$ of $\boldsymbol{x}_0$ such that, for any initial condition $\boldsymbol{x}_{10}$ in $\mathcal{W}$, the corresponding solution $\boldsymbol{x}_1(t)$ with $t \geq t_0$ of differential equation [8.9] belongs to the neighborhood $\mathcal{V}$.

It can be written that $\boldsymbol{x}_0$ is a Lyapunov-stable equilibrium position of system [8.9] if and only if

$$\forall\, \varepsilon \in \mathbb{R}^{+*}, \exists\, \eta \in \mathbb{R}^{+*}, \forall\, \boldsymbol{x}_{10},\ \|\boldsymbol{x}_{10} - \boldsymbol{x}_0\| \leq \eta \Longrightarrow \forall\, t,\ t \geq t_0,\ \|\boldsymbol{x}_1(t) - \boldsymbol{x}_0\| \leq \varepsilon.$$

where $\boldsymbol{x}_1(t)$ denotes the solution of system [8.9] such that $\boldsymbol{x}_1(t_0) = \boldsymbol{x}_{10}$.

Let us note that in the Weierstrass discussion, we have given a weaker definition of stability associated only with $\boldsymbol{Q}$ and not with $(\boldsymbol{Q}, \boldsymbol{P})$, which represents $\boldsymbol{x}$ in the system of differential equations [8.9].

DEFINITION 8.5.– ***The asymptotic stability***. An equilibrium position $\boldsymbol{x}_0$ of system [8.9] is asymptotically stable if and only if $\boldsymbol{x}_0$ is Lyapunov stable and if there exists a neighborhood $\mathcal{V}$ of $\boldsymbol{x}_0$ such that for any initial condition $\boldsymbol{x}_{10}$ in $\mathcal{V}$, the associated solution $\boldsymbol{x}_1(t)$ of system [8.9] satisfies

$$\lim_{t \longrightarrow +\infty} \boldsymbol{x}_1(t) = \boldsymbol{x}_0.$$

So, the position $\boldsymbol{x}_0$ is asymptotically stable if and only if $\boldsymbol{x}_0$ is Lyapunov stable and if

$$\lim_{t \longrightarrow +\infty} \|\boldsymbol{x}_1(t) - \boldsymbol{x}_0\| = 0.$$

Let us note that definitions require the existence of a norm in the phase space. Nonetheless, neither the Lyapunov stability nor the asymptotic stability depend on the choice of the norm because all norms are equivalent in $\mathbb{R}^n$.

8.5. A necessary condition of stability

LEMMA 8.1.– Given a positive definite quadratic form

$$\sum_{i,j=1}^{n} b_{ij} x_i x_j.$$

– For any strictly positive real ε_1, there exists a strictly positive real η_1 such that

$$\sum_{i,j=1}^{n} b_{ij}x_i x_j \leq \eta_1 \quad \Longrightarrow \quad \sup_i |x_i| \leq \varepsilon_1.$$

– For any strictly positive real ε_2, there exists a strictly positive real η_2 such that

$$\sup_i |x_i| \leq \eta_2 \quad \Longrightarrow \quad \sum_{i,\,j=1}^{n} b_{ij}x_i x_j \leq \varepsilon_2.$$

PROOF.– There are two parts in the lemma.

The proof of the second property results from the continuity of a second-degree polynomial of n variables $x_i,\ i \in \{1, \cdots, n\}$ at $(0, \cdots, 0)$.

Proof for the first property:

Let $x_i = \sum_{j=1}^{n} \alpha_{ij} y_j,\ i \in \{1, \cdots, n\}$ be a change of coordinates $(\det\{\alpha_{ij}\} \neq 0)$ such that in the new basis

$$\sum_{i,\,j=1}^{n} b_{ij}x_i x_j = \sum_{i=1}^{n} \lambda_i^2 y_i^2.$$

All $\lambda_i,\ i \in \{1, \cdots, n\}$ are non-zero. Let us denote

$$\lambda^2 = \inf_i \lambda_i^2,$$

then,

$$\sum_{i=1}^{n} \lambda_i^2 y_i^2 \leq \eta_1 \quad \Longrightarrow \quad \forall\, i,\ i \in \{1, \cdots, n\},\ y_i^2 \leq \frac{\eta_1}{\lambda_i^2} \quad \Longrightarrow$$

$$\forall\, i,\ i \in \{1, \cdots, n\},\ |y_i| \leq \frac{\sqrt{\eta_1}}{|\lambda|} \quad \Longrightarrow \quad \sup_i |y_i| \leq \frac{\sqrt{\eta_1}}{|\lambda|}.$$

Furthermore,

$$|x_i| \leq \left| \sum_{j=1}^{n} \alpha_{ij} y_j \right| \Longrightarrow |x_i| \leq \sup_j |y_j| \left[\sum_{j=1}^{n} |\alpha_{ij}| \right].$$

Let us write $M_i = \sum_{j=1}^{n} |\alpha_{ij}|$; because $\det\{\alpha_{ij}\} \neq 0$, $M_i \neq 0$. We denote $M = \sup_i M_i$, then

$$|x_i| \leq M \sup_j |y_j| \quad \Longrightarrow \quad \sup_i |xi| \leq M \sup_j |y_j| \quad \Longrightarrow \quad \sup_i |xi| \leq M \frac{\sqrt{\eta_1}}{|\lambda|}.$$

Let us choose η_1 such that $M \frac{\sqrt{\eta_1}}{|\lambda|} \leq \varepsilon_1$ or $\eta_1 \leq \frac{\varepsilon_1^2 \lambda^2}{M^2}$, then

$$\forall\, \varepsilon_1 \in \mathbb{R}^{+*}, \exists\, \eta_1 \in \mathbb{R}^{+*} \text{ such that } \sum_{i,\ j=1}^{n} b_{ij} x_i x_j \leq \eta_1 \Longrightarrow \sup_i |xi| \leq \varepsilon_1,$$

which demonstrates the first property. ■

THEOREM 8.2.– ***The Lejeune Dirichlet theorem***. Given a Lagrangian dynamical system depending on n parameters $\boldsymbol{Q} = [q_1, \cdots, q_n]^*$, we assume that:

– The kinetic energy T is minored by a positive definite quadratic form, i.e. when $\boldsymbol{P} = [p_1, \cdots, p_n]^*$ denotes the conjugate variables of $\boldsymbol{Q}$ and

$$T = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij}\,(q_1, \cdots, q_n)\, p_i\, p_j,$$

there exists a quadratic form with constant coefficients b_{ij} such that for any $\boldsymbol{Q}$,

$$\frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij}\,(q_1, \cdots, q_n)\, p_i\, p_j \geq \sum_{i,j=1}^{n} b_{ij} p_i\, p_j.$$

– The potential energy W has a strict local minimum at

$$\boldsymbol{Q}_0 = [q_{10}, \cdots, q_{n0}]^*.$$

Then, $(\boldsymbol{P} = \boldsymbol{0},\ \boldsymbol{Q} = \boldsymbol{Q}_0)$ is a Lyapunov-stable equilibrium position.

PROOF.– We are in the case of energy first-integral E and $\boldsymbol{Q}_0$ is a critical point of the potential energy W. Making translations in the space of parameters, we assume $\boldsymbol{Q}_0 = \boldsymbol{0}$ and $W(\boldsymbol{0}) = 0$. We write

$$E\,(\boldsymbol{P}, \boldsymbol{Q}) = T\,(\boldsymbol{P}, \boldsymbol{Q}) + W\,(\boldsymbol{Q}),$$

Then, $E(\boldsymbol{P},\boldsymbol{Q}) = E(\boldsymbol{P}_0,\boldsymbol{Q}_0)$, where $(\boldsymbol{P}_0,\boldsymbol{Q}_0)$ denotes the initial conditions. We denote

$$\|\boldsymbol{Q}\| = \sup_i |q_i| \quad \text{and} \quad \|\boldsymbol{P}\| = \sup_i |p_i|$$

Preliminary 1:

At $\mathbf{0}$, potential $W(\boldsymbol{Q})$ is strict local minimum. Therefore, there exists a strictly positive real A such that for $\|\boldsymbol{Q}\| \leq A$, $W(\boldsymbol{Q})$ has an absolute minimum at $\mathbf{0}$; this property is conserved by decreasing the value of A.

Preliminary 2:

Given $M = \inf_{\|\boldsymbol{Q}\|=A} W(\boldsymbol{Q})$. Then, $M > 0$, otherwise $W(\boldsymbol{Q})$ has no strict minimum at $\mathbf{0}$. Given $(\boldsymbol{P}_0,\boldsymbol{Q}_0)$ such that $\|\boldsymbol{Q}_0\| \leq A$ and $E(\boldsymbol{P}_0,\boldsymbol{Q}_0) < M$, with $T(\boldsymbol{P},\boldsymbol{Q})$ being positive, and for all t, $E(\boldsymbol{P},\boldsymbol{Q}) < M$, we deduce $W(\boldsymbol{Q}) < M$. Hence, for any $t \geq t_0$, $\|\boldsymbol{Q}\| \leq A$.

Indeed, on the contrary, there would exist t_1 such that $\|\boldsymbol{Q}(t_1)\| > A$ and, according to the intermediate value theorem on $[t_0,\ t_1]$, there exists t_2 such that $\|\boldsymbol{Q}(t_2)\| = A$, and we would get $W(\boldsymbol{Q}(t_2)) \geq M$, which is impossible.

Preliminary 3:

E will be assumed to be smaller than M. Thus, for any $\boldsymbol{Q}$ such that $\|\boldsymbol{Q}\| \leq A$, we have

$$\forall\, \varepsilon \in \mathbb{R}^{+*}, \exists\, \eta_1 \in \mathbb{R}^{+*} \text{ such that } \forall\, \boldsymbol{Q},\ W(\boldsymbol{Q}) \leq \eta_1 \implies \|\boldsymbol{Q}\| \leq \varepsilon.$$

Otherwise,

$$\exists\, \varepsilon_1 \in \mathbb{R}^{+*}, \forall\, n \in \mathbb{N},\ \exists\, \boldsymbol{Q}_n, \text{ such that } W(\boldsymbol{Q}_n) \leq \frac{1}{n} \text{ and } \|\boldsymbol{Q}_n\| > \varepsilon_1.$$

Because $\|\boldsymbol{Q}_n\| \leq A$, the sequence $\boldsymbol{Q}_n$ satisfies $\lim_{n\longrightarrow\infty} W(\boldsymbol{Q}_n) = 0$. Let $\boldsymbol{Q}_c$ be the limit of a convergent extracted sub-sequence. We have $\|\boldsymbol{Q}_c\| \leq A$ and $W(\boldsymbol{Q}_c) = 0$ for $\|\boldsymbol{Q}_c\| \geq \varepsilon$; this proves that the minimum is not strict.

Consequences:

Using Lemma 8.1 relative to the minorant quadratic form,

$$\forall\, \varepsilon \in \mathbb{R}^{+*},\ \exists\, \eta_1 \in \mathbb{R}^{+*} \text{ such that } T(\boldsymbol{P},\boldsymbol{Q}) \leq \eta_1 \implies \|\boldsymbol{P}\| \leq \varepsilon.$$

Using Preliminary 3,

$$\forall\, \varepsilon \in \mathbb{R}^{+*},\ \exists\, \eta_2 \in \mathbb{R}^{+*} \text{ such that } W(\boldsymbol{Q}) \leq \eta_2 \Longrightarrow \|\boldsymbol{Q}\| \leq \varepsilon.$$

Let us write $\eta = \inf(\eta_1,\ \eta_2)$, then

$$E(\boldsymbol{P}, \boldsymbol{Q}) \leq \eta \quad \Longrightarrow \quad \|\boldsymbol{P}\| \leq \varepsilon \ \text{ and } \ \|\boldsymbol{Q}\| \leq \varepsilon.$$

By continuity of $W(\boldsymbol{Q})$, there exists $\nu_1,\ \nu_1 \in \mathbb{R}^{+*}$ such that

$$\|\boldsymbol{Q}_0\| \leq \nu_1 \quad \Longrightarrow \quad W(\boldsymbol{Q}_0) \leq \frac{\eta}{2}$$

and because $\alpha_{ij},\ i, j \in \{1, \cdots, n\}$ are continuous at $\boldsymbol{Q} = \boldsymbol{0}$, the kinetic energy $T(\boldsymbol{P}, \boldsymbol{Q})$ is continuous with respect to $(\boldsymbol{P}, \boldsymbol{Q})$, and ν_1 being chosen sufficiently small, there exists $\nu_2,\ \nu_2 \in \mathbb{R}^{+*}$ such that

$$\|\boldsymbol{Q}_0\| \leq \nu_1 \quad \text{and} \quad \|\boldsymbol{P}_0\| \leq \nu_2 \quad \Longrightarrow \quad T(\boldsymbol{P}_0, \boldsymbol{Q}_0) \leq \frac{\eta}{2}.$$

The two implications are simultaneously satisfied. Let us write $\nu = \inf(\nu_1,\ \nu_2)$, then

$$\forall\, \varepsilon \in \mathbb{R}^{+*}, \quad \exists\, \eta \in \mathbb{R}^{+*} \text{ and } \exists\, \nu \in \mathbb{R}^{+*}$$

such that

$$\|\boldsymbol{Q}_0\| \leq \nu \quad \text{and} \quad \|\boldsymbol{P}_0\| \leq \nu \quad \Longrightarrow \quad \left\{ T(\boldsymbol{P}_0, \boldsymbol{Q}_0) \leq \frac{\eta}{2} \quad \text{and} \quad W(\boldsymbol{Q}_0) \leq \frac{\eta}{2} \right\}.$$

Therefore,

$$E(\boldsymbol{P}, \boldsymbol{Q}) \leq \eta \quad \Longrightarrow \quad \|\boldsymbol{P}\| \leq \varepsilon \quad \text{and} \quad \|\boldsymbol{Q}\| \leq \varepsilon,$$

which demonstrates the Lyapunov stability at $(\boldsymbol{0}, \boldsymbol{0})$. ■

REMARK 8.4.– It is important to note that the position $(\boldsymbol{P} = \boldsymbol{0}, \boldsymbol{Q}_0)$ is not asymptotically stable. Indeed, if the position is asymptotically stable, the total energy $E(\boldsymbol{P}, \boldsymbol{Q})$ would go towards 0 while t goes to infinity, and here the energy is constant in the motion and non-zero except for the position $(\boldsymbol{P} = \boldsymbol{0},\ \boldsymbol{Q}_0)$.

Let us note that for a system with one degree of freedom, an equilibrium position q_0, which is not strict local minimum of W, is unstable in the sense of Lyapunov. It seems natural to conjecture that a dynamical system with n degrees of freedom $(n > 1)$ whose equilibrium position is not strict local minimum of W is always unstable.

8.6. Linearization of a differential equation

8.6.1. *Preliminary*

An autonomous differential equation is in the form:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \qquad [8.11]$$

where

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1\left(x_{1,\cdots},x_n\right) \\ \vdots \\ f_n\left(x_{1,\cdots},x_n\right) \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We assume that equation [8.11] admits an equilibrium position at $\boldsymbol{x}_0$. The mapping $\boldsymbol{f}$ is continuously differentiable at $\boldsymbol{x}_0$. Making a translation in the space of parameters, we assume that $\boldsymbol{x}_0 = \boldsymbol{0}$ and we choose $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. In the regular case, there exists a linear mapping represented by matrix $\mathcal{A} = \dfrac{\partial \boldsymbol{f}(\boldsymbol{0})}{\partial \boldsymbol{x}}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}(\boldsymbol{0})}{\partial \boldsymbol{x}}\,\boldsymbol{x} + \|\boldsymbol{x}\|\,\boldsymbol{R}(\boldsymbol{x}) \quad \text{where} \quad \lim_{\boldsymbol{x}\longrightarrow\boldsymbol{0}} \boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{0}.$$

The matrix $\mathcal{A} = \{a_{ij}\},\ i,j \in \{1,\cdots,n\}$ with $a_{ij} = \dfrac{\partial f_i}{\partial x_j}(0,\cdots,0)$ represents this mapping.

DEFINITION 8.6.– The differential equation with constant coefficients

$$\frac{d\boldsymbol{x}}{dt} = \mathcal{A}\,\boldsymbol{x}. \qquad [8.12]$$

is called the ***linearized differential*** equation of equation [8.11] in the neighborhood of the equilibrium position $\boldsymbol{x} = \boldsymbol{0}$.

This differential equation is made up of a system of n first-order differentiable scalar equations.

REMARK 8.5.– As tangent linear mapping $\mathcal{A}$, linearization is an intrinsic operation, i.e. independent of the chosen system of coordinate. Since $\mathcal{A}$ has constant coefficients, equation [8.12] has the general solution

$$\boldsymbol{x}(t) = e^{(t-t_0)\mathcal{A}}\boldsymbol{x}(t_0)$$

where $\boldsymbol{x}(t_0)$ denotes the n initial conditions at t_0 and

$$e^{(t-t_0)\mathcal{A}} = 1 + (t-t_0)\,\mathcal{A} + \frac{(t-t_0)^2\,\mathcal{A}^2}{2!} + \cdots + \frac{(t-t_0)^n\,\mathcal{A}^n}{n!} + \cdots,$$

is a converging matrix series.

THEOREM 8.3.– If eigenvalues $\lambda_i,\ i \in \{1, \cdots, n\}$ of $\mathcal{A}$ have a strictly negative real part, the equilibrium position $\boldsymbol{x} = \mathbf{0}$ of differential equation [8.12] is asymptotically stable.

PROOF.– Two cases are possible.

First case: the matrix $\mathcal{A}$ is diagonalizable. In axes associated with eigenvectors of the complexification of vector space, we can write the general solution of the system by using scalar components in the form

$$x_i(t) = e^{\lambda_i(t-t_0)}x_i(t_0),$$

where $x_i(t_0)$ is the initial condition associated with component number i. Then,

$$|x_i(t)| \leq e^{Re(\lambda_i)(t-t_0)}\,|x_i(t_0)|\,,$$

where $Re\,(\lambda_i)$ denotes the real part of λ_i. For $Re\,(\lambda_i) < 0$, we immediately obtain that $\lim_{t\longrightarrow\infty} x_i(t) = 0$ and solution $x_i = 0$ is asymptotically stable. Let us note that if $Re\,(\lambda_i) = 0$, the solution $x_i = 0$ is Lyapunov stable.

Second case: Matrix $\mathcal{A}$ is not diagonalizable. Any $n \times n$ complex matrix is similar to a matrix of the form

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{J}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{J}_p \end{bmatrix},$$

where $p < n$ and $\mathcal{J}_i,\ i \in \{1, \cdots, p\}$ are square matrices of the form

$$\mathcal{J}_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

Since $\lambda_i,\ i \in \{1, \cdots, p\}$ are eigenvalues of $\mathcal{A}$, system [8.12] is constituted of p autonomous systems of the form,

$$\frac{d\boldsymbol{y}_i}{dt} = \mathcal{J}_i\, \boldsymbol{y}_i \quad \text{where} \quad \boldsymbol{y}_i = \begin{bmatrix} y_1 \\ \vdots \\ y_{r_i} \end{bmatrix} \quad \text{with} \quad \sum_{i=1}^{p} r_i = n. \qquad [8.13]$$

We can write

$$\mathcal{J}_i = \lambda_i\, \mathbf{1} + \mathcal{N}_i, \quad \text{then} \quad \mathcal{N}_i = \begin{bmatrix} 0 & \mathbf{1} & 0 \\ \vdots & \ddots & \mathbf{1} \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{hence} \quad (\mathcal{N}_i)^{r_i} = \mathbf{0}.$$

Therefore, $\mathcal{N}_i$ is nilpotent. The general solution associated with the partial system [8.13] is

$$\boldsymbol{y}_i(t) = e^{\lambda_i (t-t_0)} e^{(t-t_0)\mathcal{N}_i} \boldsymbol{y}_i(t_0) = e^{\lambda_i (t-t_0)} \left(\sum_{j=0}^{p-1} \frac{(t-t_0)^j}{j!} \mathcal{N}_j \right) \boldsymbol{y}_i(t_0).$$

The components of $\boldsymbol{y}_i(t)$ are in the form $e^{\lambda_i (t-t_0)} P(t)$, where $P(t)$ is a t-polynomial of degree less than n. For $Re\,(\lambda_i) < 0$, the solution has the limit 0 when t goes to infinity and the position $\mathbf{0}$ is asymptotically stable. ■

8.6.2. *Application to Lagrangian dynamical systems*

A Lagrangian dynamical system has

– a kinetic energy

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}\,(q_1, \cdots, q_n)\, \dot{q}_i\, \dot{q}_j = \frac{1}{2}\, \dot{\boldsymbol{Q}}^*\, \mathcal{A}\, \dot{\boldsymbol{Q}}$$

or in conjugate variables,

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a'_{ij}\,(q_1, \cdots, q_n)\, p_i\, p_j = \frac{1}{2}\, \boldsymbol{P}^*\, (\mathcal{A}^*)^{-1}\, \boldsymbol{P}$$

with $\quad \left\{ a'_{ij},\ i, j \in \{1, \cdots, n\} \right\} = (\mathcal{A}^*)^{-1},$

– a potential energy

$$W = W(q_1, \cdots, q_n).$$

The position $\boldsymbol{Q} = \boldsymbol{0}$ is assumed to be the equilibrium position; therefore, $\dfrac{\partial W(\boldsymbol{0})}{\partial \boldsymbol{Q}} = \boldsymbol{0}$, i.e. $\boldsymbol{Q} = \boldsymbol{0}$ is singular point of the potential energy W. The system of the equations of motion is the system of Hamilton equations,

$$\dot{\boldsymbol{P}} = -\left(\frac{\partial H}{\partial \boldsymbol{Q}}\right)^* \text{ and } \dot{\boldsymbol{Q}} = \left(\frac{\partial H}{\partial \boldsymbol{P}}\right)^* \text{ where } H(\boldsymbol{P}, \boldsymbol{Q}) = T + W. \quad [8.14]$$

Since $(\boldsymbol{P} = \boldsymbol{0}, \boldsymbol{Q} = \boldsymbol{0})$ is solution of

$$\frac{\partial H(\boldsymbol{P}, \boldsymbol{Q})}{\partial \boldsymbol{Q}} = \boldsymbol{0} \qquad \text{and} \qquad \frac{\partial H(\boldsymbol{P}, \boldsymbol{Q})}{\partial \boldsymbol{P}} = \boldsymbol{0},$$

the Taylor development of the right-hand side of [8.14] in the neighborhood of $(\boldsymbol{0}, \boldsymbol{0})$ has no constant terms. The linearized system of [8.14] has linear terms in $\boldsymbol{P}$ and $\boldsymbol{Q}$. These terms come from the quadratic terms of the second-order development of $H(\boldsymbol{P}, \boldsymbol{Q})$ at $(\boldsymbol{0}, \boldsymbol{0})$.

THEOREM 8.4.– To linearize a Lagrangian dynamical system in the neighborhood of the equilibrium position $\boldsymbol{Q} = \boldsymbol{0}$, we consider the system with the kinetic energy

$$T_2 = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij}\, \dot{q}_i\, \dot{q}_j \quad \text{with} \quad \alpha_{ij} = a_{ij}(0, \cdots, 0),\ i, j \in \{1, \cdots, n\}$$

and the potential energy

$$W_2 = \frac{1}{2} \sum_{i,j=1}^{n} b_{ij}\, q_i\, q_j \quad \text{with} \quad b_{ij} = \frac{\partial^2 W}{\partial q_i \partial q_j}(0, \cdots, 0),\ i, j \in \{1, \cdots, n\}.$$

EXAMPLE 8.4.– Consider the one degree of freedom system

$$T = \frac{1}{2}\, a(q)\, \dot{q}^2 \quad \text{and} \quad W = W(q).$$

The position $q = 0$ is assumed to be a stable equilibrium position; it is enough that $\dfrac{\partial W}{\partial q}(0) = 0$ and $\dfrac{\partial^2 W}{\partial q^2}(0) > 0$. We replace the expression for potential energy by approximating parabola,

$$T_2 = \frac{1}{2} a(0)\, \dot{q}^2 \quad \text{and} \quad W_2 = \frac{1}{2}\, b(0)\, q^2 \quad \text{where} \quad b(0) = \frac{\partial^2 W}{\partial q^2}(0).$$

The linearized system is

$$\ddot{q} + \omega^2 q = 0 \quad \text{where} \quad \omega^2 = \frac{b(0)}{a(0)}.$$

Its solution is

$$q = A\,\sin\left[\omega\,(t - t_0)\right] \quad \text{and} \quad p = A\,a(0)\,\omega\,\cos\left[\omega\,(t - t_0)\right].$$

In the phase space, the approximated trajectory is represented by ellipse

$$q^2 + \frac{p^2}{\omega^2 a(0)} = A^2,$$

The fundamental question is: what is the nature of the linearized solution? Is it a neighbor solution of the nonlinearized motion?

EXERCISE 8.1.– Study the period of "small oscillations" in the neighborhood of a stable equilibrium position of a weighted point in motion along a rigid wire situated in a vertical plane. The equation of the wire in the vertical plane Oxy is of the form $y = y(x)$. It is assumed that the stable equilibrium position corresponds to $x = 0$. The kinetic energy and potential energy are

$$T = \frac{1}{2}\,m\,\left(1 + y'(x)^2\right)\,\dot{x}^2 \text{ and } W = m\,g\,y(x) \text{ with } y'(0) = 0 \text{ and } y''(0) > 0.$$

For the sake of simplicity, we choose the units such that mass $m = 1$ and acceleration due to gravity $g = 1$. We are back to the previous example, for which

$$T_2 = \frac{1}{2}\,\dot{x}^2 \quad \text{and} \quad W_2 = \frac{1}{2}\,\omega^2\,x^2 \quad \text{with} \quad \omega^2 = y''(0).$$

The period of the small oscillations is $\mathcal{T} = \dfrac{2\,\pi}{\omega}$.

8.6.3. *Small oscillations of a Lagrangian dynamical system*

DEFINITION 8.7.– In the neighborhood of a stable equilibrium position $\boldsymbol{Q}_0$, the motions of the linearized system of motion equations are called small oscillations.

The linearized kinetic energy and potential energy of a Lagrangian dynamical system in the neighborhood of the equilibrium position $\boldsymbol{Q}_0 = \boldsymbol{0}$ are

$$T_2 = \frac{1}{2}\,\dot{\boldsymbol{Q}}^*\,\mathcal{A}\,\dot{\boldsymbol{Q}} \quad \text{and} \quad W_2 = \frac{1}{2}\,\boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q},$$

where $\mathcal{A}$ is associated with a positive definite quadratic form. The Lagrange equations of motion are

$$\frac{d}{dt}\left(\mathcal{A}\,\dot{\boldsymbol{Q}}\right) + \mathcal{B}\,\boldsymbol{Q} = \boldsymbol{0}. \qquad [8.15]$$

In an orthonormal basis of the quadratic form $\boldsymbol{X}^*\mathcal{A}\,\boldsymbol{X}$, the two quadratic forms $\boldsymbol{X}^*\mathcal{A}\,\boldsymbol{X}$ and $\boldsymbol{X}^*\mathcal{B}\,\boldsymbol{X}$ can be written

$$\boldsymbol{X}^*\mathcal{A}\,\boldsymbol{X} = \boldsymbol{Y}^*\boldsymbol{Y} \text{ and } \boldsymbol{X}^*\mathcal{B}\,\boldsymbol{X} = \sum_{i=1}^{n}\lambda_i Y_i^2 \text{ where } \boldsymbol{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix},$$

and $\boldsymbol{Y} = \mathcal{C}\,\boldsymbol{X}$ denotes the change of coordinates with passage matrix $\mathcal{C}$ associated with the new orthonormal basis with respect to $\boldsymbol{X}^*\mathcal{A}\,\boldsymbol{X}$. Let us write $\boldsymbol{Q}_1 = \mathcal{C}\,\boldsymbol{Q}$, then $\dot{\boldsymbol{Q}}_1 = \mathcal{C}\,\dot{\boldsymbol{Q}}$ and

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_{1i}^2 \quad \text{and} \quad W_2 = \frac{1}{2}\sum_{i=1}^{n}\lambda_i\, q_{1i}^2.$$

Values $\lambda_i,\ i \in \{1, \cdots, n\}$ associated with the quadratic form $\mathcal{B}$ for the $\mathcal{A}$-metric are the eigenvalues of $\mathcal{B}$ with respect to $\mathcal{A}$.

THEOREM 8.5.– The eigenvalues of $\mathcal{B}$ with respect to $\mathcal{A}$ are the solutions of the characteristic equation

$$\det\left(\mathcal{B} - \lambda\mathcal{A}\right) = 0,$$

where all roots of the characteristic equation are real.

PROOF.– In the new basis, " ˜ " denotes the new different tensors.

$$\tilde{\mathcal{B}} = \left(\mathcal{C}^{-1}\right)^*\mathcal{B}\,\mathcal{C}^{-1} \quad \text{then} \quad \tilde{\mathcal{B}} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{1} = \left(\mathcal{C}^{-1}\right)^*\mathcal{A}\,\mathcal{C}^{-1}.$$

The eigenvectors of $\tilde{\mathcal{B}}$ corresponding to eigenvalues $\lambda_1, \cdots, \lambda_n$ satisfy $\left(\tilde{\mathcal{B}} - \lambda\boldsymbol{1}\right)\tilde{\boldsymbol{V}} = \boldsymbol{0}$, where $\tilde{\boldsymbol{V}} \neq \boldsymbol{0}$. Vectors $\tilde{\boldsymbol{V}}$ are the new eigenvectors. Let us write $\boldsymbol{V} = \mathcal{C}^{-1}\tilde{\boldsymbol{V}}$, then $\mathcal{C}^*\left(\tilde{\mathcal{B}} - \lambda\boldsymbol{1}\right)\mathcal{C}\boldsymbol{V} = \boldsymbol{0}$ and, consequently, $\left(\mathcal{B} - \lambda\mathcal{A}\right)\boldsymbol{V} = \boldsymbol{0}$. Thus, $\boldsymbol{V}$ is the eigenvector of $\mathcal{B}$ with respect to $\mathcal{A}$ associated with the eigenvalue $\lambda \in \{\lambda_1, \cdots, \lambda_n\}$ if and only if $\det\left(\mathcal{B} - \lambda\mathcal{A}\right) = 0$. The eigenvalues are real because the potential energy is real in any basis and for all values of $q_{1i}^2,\ i \in \{1, \cdots, n\}$. ■

In the new coordinate system, the Lagrange equations are written as

$$\frac{d^2\boldsymbol{Q}_1}{dt^2} + \tilde{\mathcal{B}}\,\boldsymbol{Q}_1 = 0.$$

In coordinates

$$\ddot{q}_{1i} + \lambda_i q_{1i} = 0, \quad \text{for } i \in \{1, \cdots, n\}.$$

We avoid using the additional index 1 and write

$$\ddot{q}_i + \lambda_i q_i = 0, \quad \text{for } i \in \{1, \cdots, n\}. \tag{8.16}$$

COROLLARY 8.1.– The small oscillations of a Lagrangian dynamical system are obtained by considering the Cartesian product of the small oscillations of the n one-dimensional differential equation system. These differential equations correspond to the eigenvectors that are orthogonal in the sense of the scalar product defined by the quadratic form associated with the linearized kinetic energy.

For each scalar differential equation, we can distinguish three cases:

a) The eigenvalue $\lambda_i = \omega_i^2 > 0$; the general solution is

$$q_i = c_i \cos\left[\omega_i \left(t - t_0\right)\right].$$

The potential energy admits a strict minimum in direction $q_i = 0$. The position $q_i = 0$ is stable for the nonlinearized equation.

b) The eigenvalue $\lambda_i = 0$; the general solution is

$$q_i = c_{i1} t + c_{i2}.$$

The position $q_i = 0$ is an unstable equilibrium position of the linearized equation. We cannot draw any conclusions for the nonlinearized system.

c) The eigenvalue $\lambda_i = -k_i^2 < 0$; the general solution is

$$q_i = c_i \cosh\left[k_i \left(t - t_0\right)\right].$$

The position $q_i = 0$ is unstable for the linearized equation. It can be shown that it is also unstable for the nonlinearized equation.

COROLLARY 8.2.– For all strictly positive eigenvalues λ_i, $i \in \{1, \cdots, n\}$, system [8.15] associates small oscillations

$$\boldsymbol{Q}(t) = \sum_{i=1}^{n} (c_{i1} \cos \omega_i t + c_{i2} \sin \omega_i t)\, \boldsymbol{V}_i \qquad [8.17]$$

where $\omega_i^2 = \lambda_i$ and $\boldsymbol{V}_i$ are the eigenvectors associated with the eigenvalues λ_i corresponding to $\mathcal{B}\,\boldsymbol{V}_i = \lambda_i\, \mathcal{A}\,\boldsymbol{V}_i$.

Indeed, the differential equation associated with the coordinate i in the basis $\boldsymbol{V}_i$, $i \in \{1, \cdots, n\}$ is

$$\ddot{q}_i + \lambda_i q_i = 0$$

whose general solution is

$$c_{i1} \cos \omega_i t + c_{i2} \sin \omega_i t. \qquad [8.18]$$

The oscillations given by [8.17] are the Cartesian product of motion $q_i(t)$ associated with direction $\boldsymbol{V}_i$ and the $n-1$ trivial motions $q_j(t) = 0$, $j \neq i$ associated with the $n-1$ other directions $\boldsymbol{V}_j$.

A motion [8.18] is called an eigenoscillation with eigenfrequencies ω_i of the Lagrangian dynamical system. The number of linearly independent eigenoscillations is equal to Sylvester's inertia index of the quadratic form $\boldsymbol{X}^* \mathcal{B}\, \boldsymbol{X}$. According to the Sylvester theorem, the index does not depend on chosen axes. In case where all eigenvalues are strictly positive, all small oscillations of [8.15] are *sum* of eigenoscillations. As periods ω_i, $i \in \{1, \cdots, n\}$ are not necessarily commensurable, this *sum* is not necessarily periodic; this is the case of the well-known Lissajous curves.

CONCLUSION 8.1.– When all the eigenvalues are strictly positive, in order to find the small oscillations of a Lagrangian dynamical system associated with [8.15], we project the motion initial conditions on the system of eigenvectors $\boldsymbol{V}_k$, $k \in \{1, \cdots, n\}$. We are back to n scalar differential equations of the form [8.16].

In practice, we look for solutions of motion in the complex form

$$\boldsymbol{Q}(t) = e^{i\omega t} \boldsymbol{V}$$

and by writing $\boldsymbol{Q}(t)$ in Lagrange equations [8.15], we obtain

$$\left(\mathcal{B} - \omega^2 \mathcal{A}\right) \boldsymbol{V} = 0$$

and we get the n eigenvalues λ_k, $k \in \{1, \cdots, n\}$. When all λ_k, $k \in \{1, \cdots, n\}$ are strictly positive, $\omega_k^2 = \lambda_k$, $k \in \{1, \cdots, n\}$, and for the n eigenvectors $\boldsymbol{V}_k$, $k \in \{1, \cdots, n\}$ that constitute an orthogonal system in $\mathcal{A}$-sense, we can write

$$\boldsymbol{Q}(t) = \mathrm{Re}\left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{C_k}\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{k}(\mathrm{t}-\mathrm{t_k})} \boldsymbol{V}_\mathrm{k}\right).$$

The result is valid even for multiple eigenvalues.

8.7. Behavior of eigenfrequencies

8.7.1. *Preliminaries*

A surface of equation $\boldsymbol{Q}^* \mathcal{B}\, \boldsymbol{Q} = 1$, where $\boldsymbol{Q} \in \mathbb{R}^n$ and $\mathcal{B}$ is a positive definite symmetric matrix, is called ellipsoid of $\mathbb{R}^n$. The half-axes' lengths of ellipsoid $\mathcal{E}$ are denoted by $a_i, i \in \{1, \cdots, n\}$ with $a_1 \geq \cdots \geq a_n$.

THEOREM 8.6.– Each section of ellipsoid $\mathcal{E}$ by a k-dimensional hyperplane $\mathcal{H}_k$ (where $k < n$) has its smallest half-axis of length shorter than or equal to a_k and

$$a_k = \max_{\mathcal{H}_k} \min_{\boldsymbol{x} \in \mathcal{H}_k \cap \mathcal{E}} \|\boldsymbol{x}\|.$$

PROOF.– Let $\mathcal{H}_{n-k+1}$ be the vector subspace of $\mathbb{R}^n$ generated by the directions of half-axes associated with $a_k, \cdots, a_n$ of the ellipsoid. The subspace dimension is $n - k + 1$. It has common points with $\mathcal{H}_k$ since $n - k + 1 + k = n + 1 > n$. Let $\boldsymbol{x}$ be an intersection point of the two vector subspaces corresponding to one direction of the intersection that meets $\mathcal{E}$. Since $\boldsymbol{x} \in \mathcal{H}_{n-k+1}$, we have $\|\boldsymbol{x}\| \leq a_k$. As $\|\boldsymbol{x}\|$ is greater than or equal to the length of the smallest half-axis of the ellipsoid $E \cap \mathcal{H}_k$, this small half-axis is not greater than a_k. Furthermore, the upper bound corresponding to the largest of the small half-axes of ellipsoid $\mathcal{E} \cap H_k$ is reached for the subspace generated by the half-axes associated with $a_1, \cdots, a_k$, where $a_1 \geq \cdots \geq a_k$ and has the value a_k. ■

8.7.2. *Behavior of eigenfrequencies with the system rigidity*

Let us consider a Lagrangian dynamical system whose kinetic energy and potential energy are

$$T = \frac{1}{2}\, \dot{\boldsymbol{Q}}^*\, \mathcal{A}\, \dot{\boldsymbol{Q}} \quad \text{and} \quad W = \frac{1}{2}\, \boldsymbol{Q}^* \mathcal{B}\, \boldsymbol{Q}.$$

It is assumed that for all $\dot{\boldsymbol{Q}} \neq \boldsymbol{0}$, the kinetic energy T is positive, and for all $\boldsymbol{Q} \neq \boldsymbol{0}$, the potential energy is also positive. The eigenvalues of $\mathcal{B}$ with respect to $\mathcal{A}$ are positive real and the system has n periodic oscillations.

DEFINITION 8.8.– For given kinetic energy $T = \frac{1}{2}\,\dot{\boldsymbol{Q}}^*\,\mathcal{A}\,\dot{\boldsymbol{Q}}$, Lagrangian dynamical system $\mathcal{S}'$ with the potential energy $W' = \frac{1}{2}\,\boldsymbol{Q}^*\mathcal{B}'\,\boldsymbol{Q}$ is said to be more rigid than Lagrangian dynamical system $\mathcal{S}$ with potential energy $W = \frac{1}{2}\,\boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q}$ if and only if

$$\boldsymbol{Q}^*\mathcal{B}'\,\boldsymbol{Q} \geq \boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q}.$$

THEOREM 8.7.– When the rigidity of a Lagrangian dynamical system increases, all eigenfrequencies increase, i.e. if $\omega_1 \leq \cdots \leq \omega_n$ are the eigenfrequencies of system $\mathcal{S}$ and $\omega'_1 \leq \cdots \leq \omega'_n$ are eigenfrequencies of the Lagrangian dynamical system $\mathcal{S}'$, the Lagrangian dynamical system being more rigid than system $\mathcal{S}$ implies

$$\omega_1 \leq \omega'_1,\ \omega_2 \leq \omega'_2, \cdots, \omega_n \leq \omega'_n.$$

If we consider the Euclidean structure defined by the kinetic energy, it may be assumed that $\mathcal{A} = \mathbf{1}$. We consider ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ defined by $\boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q} = 1$ and ellipsoid $\mathcal{E}'$ in $\mathbb{R}^n$ defined by $\boldsymbol{Q}^*\mathcal{B}'\,\boldsymbol{Q} = 1$.

LEMMA 8.2.– If Lagrangian dynamical system $\mathcal{S}'$ is more rigid than Lagrangian dynamical system $\mathcal{S}$, then $\mathcal{E}'$ is contained inside $\mathcal{E}$.

Indeed, $\boldsymbol{Q}^*\mathcal{B}'\,\boldsymbol{Q} \geq \boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q}$; if $\boldsymbol{Q}^*\mathcal{B}\,\boldsymbol{Q} = 1$, then $\boldsymbol{Q}^*\mathcal{B}'\,\boldsymbol{Q} \geq 1$ and points $\boldsymbol{Q}$ are on ellipsoid $\mathcal{E}$ and outside ellipsoid $\mathcal{E}'$.

LEMMA 8.3.– The lengths of half-axes of ellipsoid $\mathcal{E}$ are inversely proportional to the eigenfrequencies, i.e. $a_i = \dfrac{1}{\omega_i},\ i \in \{1, \cdots, n\}$.

With the kinetic energy being $T = \frac{1}{2}\,\dot{\boldsymbol{Q}}^*\,\dot{\boldsymbol{Q}}$, in an orthonormal basis, the matrix $\mathcal{B}$ is diagonal and

$$\sum_{i=1}^{n} \lambda_i\, q_i^2 = 1 \quad \Longrightarrow \quad \forall i,\ \ \lambda_i\, a_i^2 = 1 \quad \Longleftrightarrow \quad a_i = \frac{1}{\sqrt{\lambda_i}} = \frac{1}{\omega_i}\,,$$

Theorem 8.7 is equivalent to:

THEOREM 8.8.– If ellipsoid $\mathcal{E}$ with half-axes $a_1, \cdots, a_n$, where $a_1 \geq \cdots \geq a_n$, contains ellipsoid $\mathcal{E}'$ with same center and with half-axes $a'_1, \cdots, a'_n$, where $a'_1 \geq \cdots \geq a'_n$, then the half-axes of ellipsoid $\mathcal{E}'$ are shorter than the half-axes of ellipsoid $\mathcal{E}$, i.e.

$$a_1 \geq a'_1,\ a_2 \geq a'_2, \cdots, a_n \geq a'_n.$$

PROOF.– Indeed, the smallest half-axis of each section by an hyperplane $\mathcal{H}_k$ of dimension k with ellipsoid $\mathcal{E}'$ is not greater than the half-axis of the section made in ellipsoid $\mathcal{E}$ by the same hyperplane of dimension k since $\mathcal{E}' \subset \mathcal{E}$. Thus,

$$\min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}'} \|\boldsymbol{x}\| \leq \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}} \|\boldsymbol{x}\|\,.$$

According to Theorem 8.6,

$$\min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}'} \|\boldsymbol{x}\| \leq \max_{\mathcal{H}_k} \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}} \|\boldsymbol{x}\| = a_k \quad \Longrightarrow \quad \max_{\mathcal{H}_k} \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}'} \|\boldsymbol{x}\| \leq a_k.$$

Furthermore, according the conclusion of Theorem 8.6 $a'_k \leq a_k$ because $a'_k = \max_{\mathcal{H}_k} \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}'} \|\boldsymbol{x}\|$. ■

REMARK 8.6.– Because the potential energy is a positive definite quadratic form, we can return to section 8.7.2 by replacing "kinetic energy" with "potential energy". If the potential energy of a Lagrangian dynamical system is unchanged, then an increase in kinetic energy (for instance, by increasing the masses) implies a decrease of each eigenfrequency. Let us note that a cracked bell has a rigidity smaller than the corresponding intact bell; consequently, sounds of the cracked bell are deeper than sounds of the intact bell.

8.7.3. *The frequencies' behavior when parameters are linked by constraints*

Consider the Lagrangian dynamical system $\mathcal{S}_n$ with n degrees of freedom. The kinetic energy and potential energy are

$$T = \frac{1}{2}\,\dot{\boldsymbol{Q}}^*\,\dot{\boldsymbol{Q}} \quad \text{and} \quad W = \frac{1}{2}\,\boldsymbol{Q}^* B\,\boldsymbol{Q} \quad \text{where} \quad \boldsymbol{Q} \in \mathbb{R}^n,$$

which correspond to periodic small oscillations around the stable equilibrium position $\mathbf{0} \in \mathbb{R}^n$. Let $\mathcal{H}_{n-1}$ denote a $n-1$-dimensional vector subspace of $\mathbb{R}^n$. Consider a Lagrangian dynamical system $\mathcal{S}_{n-1}$ with $n-1$ degrees of freedom associated with $\mathcal{H}_{n-1}$ when writing the nullity of the linear relation between parameters $(q_1, \cdots, q_n)$. Such a combination corresponds to the linearized part in the neighborhood of $(0, \cdots, 0)$ of a holonomic constraint related to $(q_1, \cdots, q_n)$. The kinetic and potential energies of $\mathcal{S}_{n-1}$ are equal to the restriction of T and W to $\mathcal{H}_{n-1}$. System $\mathcal{S}_{n-1}$ is deduced from initial system $\mathcal{S}_n$ submitted to the linear constraint. Assume that $\mathcal{S}_n$ has n eigenfrequencies, $\omega_1, \cdots, \omega_n$ such that $\omega_1 \leq \cdots \leq \omega_n$ and $\mathcal{S}_{n-1}$ has $n-1$ eigenfrequencies $\omega'_1, \cdots, \omega'_{n-1}$ such that $\omega'_1 \leq \cdots \leq \omega'_{n-1}$.

THEOREM 8.9.– The eigenfrequencies of Lagrangian dynamical system $\mathcal{S}_{n-1}$ associated with a holonomic constraint separate the eigenfrequencies of the initial Lagrangian dynamical system $\mathcal{S}$ as follows:

$$\omega_1 \leq \omega'_1 \leq \omega_2 \leq \omega'_2 \leq \cdots \omega_{n-1} \leq \omega'_{n-1} \leq \omega_n.$$

Because of Lemma 8.3 this theorem is equivalent to:

THEOREM 8.10.– Consider the section $\mathcal{E}_{n-1}$ of ellipsoid $\mathcal{E}_n$ of half-axes' lengths $a_1, \cdots, a_n$, where $a_1 \geq \cdots \geq a_n$, by an $n-1$-dimensional hyperplane. Then, the half-axes $a'_1, \cdots, a'_{n-1}$, where $a'_1 \geq \cdots \geq a'_{n-1}$ of section $\mathcal{E}_{n-1}$ separates the half-axes of $\mathcal{E}_n$ as

$$a_1 \geq a'_1 \geq a_2 \geq a'_2 \geq \cdots a_{n-1} \geq a'_{n-1} \geq a_n.$$

PROOF.– Indeed, let $\mathcal{H}'_k$ be a k-dimensional hyperplane included in $\mathcal{H}_{n-1}$. Since $\mathcal{H}'_k \cap \mathcal{E}_{n-1} = \mathcal{H}'_k \cap \mathcal{E}_n$, we get

$$a'_k = \max_{\mathcal{H}'_k} \min_{\boldsymbol{x} \in \mathcal{H}'_k \cap \mathcal{E}_{n-1}} \|\boldsymbol{x}\| = \max_{\mathcal{H}'_k} \min_{\boldsymbol{x} \in \mathcal{H}'_k \cap \mathcal{E}_n} \|\boldsymbol{x}\| .$$

However, with $\mathcal{H}_k$ being any k-dimensional hyperplane and since we consider more elements in $\mathcal{H}_k$,

$$\max_{\mathcal{H}'_k} \min_{\boldsymbol{x} \in \mathcal{H}'_k \cap \mathcal{E}_n} \|\boldsymbol{x}\| \leq \max_{\mathcal{H}_k} \min_{\boldsymbol{x} \in \mathcal{H}_k \cap \mathcal{E}_n} \|\boldsymbol{x}\| .$$

Furthermore, according to Theorem 8.6

$$\max_{\mathcal{H}_k} \min_{\boldsymbol{x} \in \mathcal{H}_k \cap \mathcal{E}_n} \|\boldsymbol{x}\| = a_k.$$

Therefore, $a_k \geq a'_k$.

We now have to prove $a'_k \geq a_{k+1}$. Let us intersect hyperplane $\mathcal{H}_{n-1}$ containing $\mathcal{E}_{n-1}$ with a $k+1$-dimensional hyperplane $\mathcal{H}_{k+1}$. The section has $k+1$ or k dimensions, depending on whether $\mathcal{H}_{n-1}$ contains $\mathcal{H}_{k+1}$ or not. The shortest half-axis of ellipsoid $\mathcal{E}_{n-1} \cap \mathcal{H}_{k+1}$ is greater than or equal to the shortest half-axis of ellipsoid $\mathcal{E}_n \cap \mathcal{H}_{k+1}$. We have

$$\min_{\boldsymbol{x} \in \mathcal{H}_{k+1} \cap \mathcal{E}_{n-1}} \|\boldsymbol{x}\| \geq \min_{\boldsymbol{x} \in \mathcal{H}_{k+1} \cap \mathcal{E}_n} \|\boldsymbol{x}\| , \quad \text{which implies}$$

$$\max_{\mathcal{H}_{k+1}} \min_{\boldsymbol{x} \in \mathcal{H}_{k+1} \cap \mathcal{E}_{n-1}} \|\boldsymbol{x}\| \geq \max_{\mathcal{H}_{k+1}} \min_{\boldsymbol{x} \in \mathcal{H}_{k+1} \cap \mathcal{E}_n} \|\boldsymbol{x}\| = a_{k+1}. \qquad [8.19]$$

Let us estimate $\max\limits_{\mathcal{H}_{k+1}} \min\limits_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\|$.

– Case (a): it corresponds to $\mathcal{H}_{k+1} \not\subseteq \mathcal{H}_{n-1}$, then $\mathcal{H}_k$ is the trace of $\mathcal{H}_{k+1}$ on $\mathcal{H}_{n-1}$; $\mathcal{H}_k$ is any hyperplane included in $\mathcal{H}_{n-1}$ and we get

$$\min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\|$$

and then

$$\max_{\mathcal{H}_k\subset\mathcal{H}_{n-1}} \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = a'_k$$

– Case (b): it corresponds to $\mathcal{H}_{k+1} \subset \mathcal{H}_{n-1}$.

$$\min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = \min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| \quad \text{with} \quad \mathcal{H}_{k+1}\subset\mathcal{H}_{n-1}$$

and thus

$$\max_{\mathcal{H}_{k+1}\subset\mathcal{H}_{n-1}} \min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = a'_{k+1}.$$

Moreover,

$$\max_{\mathcal{H}_{k+1}} \min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = \sup\left(\max_{\mathcal{H}_k\subset H_{n-1}} \min_{\boldsymbol{x}\in\mathcal{H}_k\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\|\right), \text{ where } \mathcal{H}_k\subset\mathcal{H}_{n-1},$$

yields

$$\max_{\mathcal{H}_{k+1}\subset\mathcal{H}_{n-1}} \min_{\boldsymbol{x}\in\mathcal{H}_{k+1}\cap\mathcal{E}_{n-1}} \|\boldsymbol{x}\| = \sup(a'_k, a'_{k+1}) = a'_k, \text{ where } \mathcal{H}_{k+1}\subset\mathcal{H}_{n-1}.$$

The half-axis of ellipsoid $\mathcal{E}_{n-1}\cap\mathcal{H}_{k+1}$ is greater than or equal to the small half-axis of ellipsoid $\mathcal{E}_n\cap\mathcal{H}_{k+1}$, then $a'_k \geq a_{k+1}$; we finally obtain relation [8.19]

$$a_k \geq a'_k \geq a_{k+1},$$

which proves the theorem. ■

8.8. Perturbed equation associated with linear differential equation

A linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ is represented by a matrix $\mathcal{A}$ and $\boldsymbol{x}$ is an element of $\mathbb{R}^n$.

DEFINITION 8.9.– The perturbed equation of the linear differential equation

$$\frac{d\boldsymbol{x}}{dt} = \mathcal{A}\,\boldsymbol{x}, \qquad [8.20]$$

in the neighborhood of the position $\boldsymbol{x} = \mathbf{0}$ is the differential equation

$$\frac{d\boldsymbol{x}}{dt} = \mathcal{A}\,\boldsymbol{x} + p(\boldsymbol{x}), \qquad [8.21]$$

where p denotes a C^1-mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $p(\mathbf{0}) = \mathbf{0}$ and $\dfrac{\partial p}{\partial \boldsymbol{x}}(\mathbf{0}) = \mathbf{0}$.

EXAMPLE 8.5.– Consider the scalar differential equation

$$\frac{dx}{dt} = -x + x^2. \qquad [8.22]$$

The associated linearized differential equation is

$$\frac{dx}{dt} = -x. \qquad [8.23]$$

– The general solution of [8.23] is

$$\varphi(t) = a\,e^{-t}.$$

where a is real constant. The solution with initial value x_0 for $t = 0$ is

$$\varphi_{x_0}(t) = x_0\,e^{-t}.$$

– The general solution of [8.22] is

$$\left|\frac{x-1}{x}\right| = |\lambda|\ e^t.$$

where λ is real constant. In the case $x \in]0,1[$, $x - 1 = \lambda\,x\,e^t$, where $\lambda < 0$ and $x = \dfrac{1}{1-\lambda\,e^t}$. Taking $x_0 \in]0,1[$ as the initial position at $t = 0$, $x_0 = \dfrac{1}{1-\lambda}$ and the solution is

$$\psi_{x_0}(t) = \frac{1}{1 + \left(\dfrac{1}{x_0} - 1\right) e^t} \equiv \frac{x_0}{x_0 + (1 - x_0)\,e^t},$$

which belongs to $]0,1[$ for $t > 0$.

It must be noted that the solutions of [8.22] and [8.23] are asymptotically stable at $x = 0$.

Let us calculate $\Delta_{x_0}(t) = \psi_{x_0}(t) - \varphi_{x_0}(t)$, then

$$\Delta_{x_0}(t) = \frac{x_0\,(1 - e^{-t})}{1 + \left(\dfrac{1}{x_0} - 1\right) e^{t}}$$

and for $t > 0$, $0 \leq \Delta_{x_0}(t) \leq x_0$.

Consequently, $\psi_{x_0}(t) - \varphi_{x_0}(t)$ is small with x_0 and goes to zero independently of x_0 when t goes to infinity.

We can ask the following questions:

– For the same initial positions, what can we deduce for solutions of differential equations [8.20] and [8.21]? Are they close in a sense to be defined?

– If $x = 0$ is a stable equilibrium position of [8.20], is it the same for [8.21]?

Let us study an estimation that gives the difference between solutions of the two differential equations and look for a theorem that answers the above questions.

CONCLUSION 8.2.– Estimation over a finite time interval of difference between solutions of differential equations [8.20] and [8.21] for the same initial condition in the neighborhood of a stable equilibrium position of the nonlinearized differential equation [8.21].

We choose the initial condition at $\mathbf{0}$. Let $\boldsymbol{z}(t)$ denote the difference between the two solutions $x(t)$ and $y(t)$ of differential equations [8.21] and [8.20], respectively; the initial condition for $\boldsymbol{z}(t)$ is $\boldsymbol{z}(0) = \mathbf{0}$. From [8.20] and [8.21], we obtain

$$\frac{d\boldsymbol{z}}{dt} = \mathcal{A}\,\boldsymbol{z} + p(\boldsymbol{x}) \;\text{ with }\; \boldsymbol{z}(0) = \mathbf{0}.$$

We deduce

$$\frac{d}{dt}\left(e^{-\mathcal{A}\,t}\boldsymbol{z}\right) = e^{-\mathcal{A}\,t}p(\boldsymbol{x}).$$

Using the finite increment theorem to the i^{th} element of the solution, and by noting that $\boldsymbol{z}(0) = \mathbf{0}$,

$$\forall\, t,\, t \in \mathbb{R}^{+*}, \exists\, t_i,\, i \in \{1, \cdots, n\},\, t_i \in]0, t[,$$

such that

$$\left[e^{-\mathcal{A}\,t}\boldsymbol{z}(t)\right]_i = t\left[e^{-\mathcal{A}\,t_i}p(\boldsymbol{x}(t_i))\right]_i.$$

We deduce

$$[\boldsymbol{z}(t)]_i = t\left[e^{\mathcal{A}(t-t_i)}p(\boldsymbol{x}(t_i))\right]_i \quad \text{hence} \quad \|\boldsymbol{z}(t)\| \leq t\sup_{t_i}\left\|e^{\mathcal{A}(t-t_i)}\right\|\|p(\boldsymbol{x}(t_i)\|.$$

Over time interval $[0,T]$, we compare the solutions. Term $\left\|e^{\mathcal{A}\,t}\right\|$ being bounded over $[0,T]$,

$$\forall\, t,\ t\in[0,T]\,,\exists\, m(T),\ m(T)\in\mathbb{R}^{+*},\ \text{such that}\ \left\|e^{\mathcal{A}\,t}\right\|\leq m(T).$$

Furthermore, $t-t_i\in[0,T]$ implies $\sup_{t_i}\left\|e^{\mathcal{A}(t-t_i)}\right\|\leq m(T)$.

As p is continuous,

$$\forall\,\varepsilon\in\mathbb{R}^{+*},\ \exists\,\eta_1\in\mathbb{R}^{+*},\ \text{such that}\ \|\boldsymbol{x}(t)\|\leq\eta_1\implies\|p(\boldsymbol{x}(t)\|\leq\frac{\varepsilon}{m(T)}.$$

Since the position $\mathbf{0}$ is Lyapunov stable,

$$\forall\, t>0,\ \forall\,\eta\in\mathbb{R}^{+*},\ \exists\,\delta\in\mathbb{R}^{+*}\ \text{such that}\ \|\boldsymbol{x}_0\|\leq\delta\implies\|\boldsymbol{x}(t)\|\leq\eta.$$

Let us choose $\eta=\eta_1$, then

$$\forall\, t\in[0,T]\,,\ \forall\varepsilon\in\mathbb{R}^{+*},\exists\,\delta\in\mathbb{R}^{+*}\ \text{such that}\ \|\boldsymbol{x}_0\|\leq\delta\implies\|\boldsymbol{z}(t)\|\leq\varepsilon\, t.$$

When the initial condition is close to a Lyapunov-stable equilibrium position of the nonlinearized equation, solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ of the two differential equations remain neighboring over a given time interval. The approximation results from the size of $\varepsilon\, t$ and therefore from the size of the time interval.

In case of strong regularity, we can answer the questions.

THEOREM 8.11.– Let differential equation

$$\frac{d\boldsymbol{x}}{dt}=\mathcal{A}\,\boldsymbol{x}+p(\boldsymbol{x}), \qquad [8.24]$$

with $p(\mathbf{0})=\mathbf{0}$ and the eigenvalues of $\mathcal{A}$ have strictly negative real parts. Moreover, for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in the closed ball of center $\mathbf{0}$ and radius r,

$$\|p\,(\boldsymbol{x}_1)-p\,(\boldsymbol{x}_2)\|\leq k(r)\,\|\boldsymbol{x}_1-\boldsymbol{x}_2\|\,,$$

where the continuous mapping k defined over $[0, a]$ with $a > 0$ has its values in $\mathbb{R}^{+*}$ and $\lim_{r \to 0} k(r) = 0$. Then, $\boldsymbol{x} = \mathbf{0}$ is an asymptotically stable equilibrium position of differential equation [8.24][1].

PROOF.– $\psi_{\boldsymbol{x}_0}(t)$ denotes the solution of equation [8.24] such that $\psi_{\boldsymbol{x}_0}(0) = \boldsymbol{x}_0$. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be close to position $\mathbf{0}$. We write $\Delta(t) = \psi_{\boldsymbol{x}_2}(t) - \psi_{\boldsymbol{x}_1}(t)$ and $\rho(t) = \|\Delta(t)\|^2$.

$$\rho(t) = \sum_{j=1}^{n} |\delta_j(t)|^2 = \sum_{j=1}^{n} \overline{\delta_j(t)}\, \delta_j(t),$$

where $\delta_j(t)$ represents the component number j of $\Delta(t)$ (the conjugate complex of $\delta_j(t)$ is denoted by $\overline{\delta_j(t)}$). It can be verified that

$$\frac{d\rho}{dt} = 2\,\mathrm{Re}\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \overline{\delta_{\mathrm{j}}(\mathrm{t})}\, \frac{\mathrm{d}\delta_{\mathrm{j}}(\mathrm{t})}{\mathrm{dt}}\right) = 2\,\mathrm{Re}\left(\left(\overline{\Delta(\mathrm{t})}\right)^{*} \frac{\mathrm{d}\Delta(\mathrm{t})}{\mathrm{dt}}\right).$$

According to differential equation [8.24],

$$\frac{d\Delta(t)}{dt} = \frac{d\psi_{\boldsymbol{x}_2}(t)}{dt} - \frac{d\psi_{\boldsymbol{x}_1}(t)}{dt} = \mathcal{A}\,\Delta(t) + p\left(\psi_{\boldsymbol{x}_2}(t)\right) - p\left(\psi_{\boldsymbol{x}_1}(t)\right).$$

Hence,

$$\frac{d\rho}{dt} = 2\,\mathrm{Re}\left(\left(\overline{\Delta(\mathrm{t})}\right)^{*} \mathcal{A}\,\Delta(\mathrm{t})\right) + 2\,\mathrm{Re}\left\{\overline{\Delta(\mathrm{t})}^{*}\left[\mathrm{p}\left(\psi_{\boldsymbol{x}_2}(\mathrm{t})\right) - \mathrm{p}\left(\psi_{\boldsymbol{x}_1}(\mathrm{t})\right)\right]\right\}.$$

According to theorems relating to the triangulation of endomorphisms, for all $\eta \in \mathbb{R}^{+*}$, we can find a basis of $\mathbb{R}^n$ in which the mapping $\mathcal{A}$ can be written

$$\mathcal{A} = \begin{bmatrix} \lambda_1 & a_{1,1} & \cdots & a_{1,n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n-1} \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad \text{where} \quad \|a_{i,j}\| \leq \eta \ \text{ for } i \leq j.$$

1 Let us note this is the case for differential equation [8.22] of Example 8.5. Indeed, we have

$$p(x) = x^2, \quad \text{hence} \quad p(x_1) - p(x_2) = (x_1 + x_2)\,(x_1 - x_2).$$

For $|x_1| \leq r$ and $|x_2| \leq r$, $|p(x_1) - p(x_2)| \leq k(r)\,|x_1 - x_2|$ with $k(r) = 2r$. The solutions are asymptotically stable when the initial condition is close to 0.

Then,

$$\mathrm{Re}\left(\overline{\Delta(\mathrm{t})}^{*}\mathcal{A}\,\Delta(\mathrm{t})\right) \leq \mathrm{Re}\left(\sum_{\mathrm{j}=1}^{\mathrm{n}}\lambda_{\mathrm{i}}\ \left|\delta_{\mathrm{j}}(\mathrm{t})\right|^{2}+\sum_{\mathrm{j}=1}^{\mathrm{n}}\mathrm{a}_{\mathrm{i,j}}\ \overline{\delta_{\mathrm{i}}(\mathrm{t})}\,\delta_{\mathrm{j}}(\mathrm{t})\right).$$

We note that $\sum_{i<j}\overline{\delta_i(t)}\,\delta_j(t) \leq \sum_i \|\delta_i(t)\|$. Since the eigenvalues of $\mathcal{A}$ have strictly negative real parts, $\max_i \mathrm{Re}\,(\lambda_i)$ is a strictly negative given real scalar. Since $\frac{1}{2}n(n-1)\,\eta$ can be chosen to be as small as we want,

$$\mathrm{Re}\left(\overline{\Delta(\mathrm{t})}^{*}\mathcal{A}\Delta(\mathrm{t})\right) \leq \left(\max_{\mathrm{i}}\mathrm{Re}\,(\lambda_{\mathrm{i}})\right)\ \|\Delta(\mathrm{t})\|^{2}+\frac{1}{2}\mathrm{n}\,(\mathrm{n}-1))\,\eta\,\|\Delta(\mathrm{t})\|^{2}=-\mathrm{N}\rho(\mathrm{t})$$

where

$$N=-\left(\max_{i}\mathrm{Re}\,(\lambda_{\mathrm{i}})\right)-\frac{1}{2}n(n-1)\eta\in\mathbb{R}^{+*}.$$

Furthermore,

$$\begin{aligned}\mathrm{Re}\left\{\overline{\Delta(\mathrm{t})}^{*}\left[\mathrm{p}\left(\psi_{\boldsymbol{x}_2}(\mathrm{t})\right)-\mathrm{p}\left(\psi_{\boldsymbol{x}_1}(\mathrm{t})\right)\right]\right\} &\leq \|\Delta(t)\|\,\|p\left(\psi_{\boldsymbol{x}_2}(t)\right)-p\left(\psi_{\boldsymbol{x}_1}(t)\right)\| \\ &\leq k(r)\,\|\Delta(t)\|=k(r)\rho(t)\end{aligned}$$

and we obtain

$$\frac{d\rho}{dt}\leq -2\,(N-k(r))\,\rho(t).$$

Therefore,

$$\rho(t)\leq\rho(t_0)\exp\left[-2\,(N-k(r))\,t\right],$$

where

$$\rho(t_0)=\|\boldsymbol{x}_2-\boldsymbol{x}_1\| \quad \text{and} \quad \lim_{t\longrightarrow+\infty}\exp\left[-2\,(N-k(r))\,t\right]=0.$$

For $\boldsymbol{x}_1=\mathbf{0}$, we find that position $\mathbf{0}$ is asymptotically stable. ■

9

The Stability of Periodic Systems

The differential equations presented in this chapter are written in dimensionless form by using appropriate units of mass, length and time.

9.1. Position of the problem

DEFINITION 9.1.– A differential equation

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad [9.1]$$

where $\boldsymbol{f}$ is a mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, is a differential equation of the period T if and only if for all $\boldsymbol{x}$ and t,

$$\boldsymbol{f}(\boldsymbol{x}, t+T) = \boldsymbol{f}(\boldsymbol{x}, t).$$

This is the case with the *Hill differential equation*,

$$\ddot{q} = -\omega^2(t) \quad \text{with} \quad \omega(t+T) = \omega(t),$$

where q is a function from $\mathbb{R}$ to $\mathbb{R}$. Indeed, by writing $p = \dot{q}$, we obtain an equation in the form of [9.1] with $[q, p]^\star = \boldsymbol{x}$:

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\omega^2(t)\, q \end{cases}$$

This equation is the linearized equation of the simple pendulum in a gravitational field where the gravitational acceleration is periodic, for example the gravitational

field of the moon (the Hill equation is also called the *lunar equation*). This is also the differential equation of a simple pendulum whose point of attachment is subject to forced vertical oscillations such that its length periodically varies. We will see that this type of differential equation has unusual properties of stability.

9.2. Flow of a periodic differential equation

DEFINITION 9.2.– As in section 7.1 of Chapter 7, the flow is the mapping associating the solution of equation [9.1] at instant t with the initial condition $\boldsymbol{x}_0$. We write

$$\boldsymbol{x} = g_t(\boldsymbol{x}_0).$$

REMARK 9.1.– It is important to note that differential equation [9.1] is not autonomous and consequently the mappings $\{g_t,\ t \in \mathbb{R}^+\}$ do not form an additive group. Indeed, the theory of differential equations proved that a necessary and sufficient condition for mappings $\{g_t,\ t \in \mathbb{R}^+\}$ to be an additive group is that $\boldsymbol{f}$ does not depend on t.

Let us study the properties of the flow g_t for periodic equations.

THEOREM 9.1.– The mapping $\boldsymbol{f}$ is periodic of period T, then

$$\forall\, s \in \mathbb{R}^+,\ g_{s+T} = g_s \circ g_T. \qquad [9.2]$$

PROOF.– Let us denote by $\boldsymbol{x}_1 = g_T(\boldsymbol{x}_0)$ the value for $t = T$ of the solution with initial condition $\boldsymbol{x}_0$ at $t = 0$. Let us write $u = t - T$, then $\boldsymbol{X}(u) = \boldsymbol{x}(t)$ and $\boldsymbol{f}(\boldsymbol{X}, u) = \boldsymbol{f}(\boldsymbol{x}, t)$. Consequently,

$$\frac{d\boldsymbol{X}}{du} = f(\boldsymbol{X}, u). \qquad [9.3]$$

The solution of [9.3] such that for $u = 0, \boldsymbol{X} = \boldsymbol{x}_1$ is $\boldsymbol{X}(u) = g_u(\boldsymbol{x}_1)$ with $\boldsymbol{X}(u) = g_u \circ g_T(\boldsymbol{x}_0)$. Furthermore, $\boldsymbol{x}(t) = g_t(\boldsymbol{x}_0)$; therefore, $\boldsymbol{x}(t) = g_{u+T}(\boldsymbol{x}_0)$.

Finally, for all $\boldsymbol{x}_0$, $g_{u+T}(\boldsymbol{x}_0) = g_u \circ g_T(\boldsymbol{x}_0)$ and $g_{s+T} = g_s \circ g_T$. ■

We deduce

$$\forall\, n \in \mathbb{Z},\ g_{(nT)} = (g_T)^n.$$

COROLLARY 9.1.– The mappings $\left\{g_{(nT)},\ n \in \mathbb{Z}\right\}$ form a multiplicative group.

DEFINITION 9.3.– The mapping g_T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called the period mapping. We write

$$g_T = \mathcal{G} \text{ and } \mathcal{G}(\boldsymbol{x}_0) = \boldsymbol{x}(T)$$

Relation [9.2] is then written

$$g_{s+T} = g_s \circ \mathcal{G}.$$

THEOREM 9.2.– Point $\boldsymbol{x}_0$ is a fixed point of mapping $\mathcal{G}$ (i.e. $\mathcal{G}(\boldsymbol{x}_0) = \boldsymbol{x}_0$) if and only if the solution of the initial condition $\boldsymbol{x}_0$ at $t = 0$ is T-periodic.

Indeed, $g_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$ is equivalent to T period of application $\boldsymbol{x}(s) = g_s(\boldsymbol{x}_0)$.

Now, let us give definitions for periodic equations in accordance with the previous paragraphs.

DEFINITION 9.4.– ***The Lyapunov stability of periodic differential equations.*** The fixed point $\boldsymbol{x}_0$ of mapping $\mathcal{G}$ is Lyapunov stable if and only if for all neighborhood $\mathcal{V}$ of $\boldsymbol{x}_0$ there exists a neighborhood $\mathcal{W}$ of $\boldsymbol{x}_0$ such that for all initial condition $\boldsymbol{x}_{10}$ in $\mathcal{W}$ then for all $n \in \mathbb{N}$, $\mathcal{G}^n(\boldsymbol{x}_{10})$ belongs to $\mathcal{V}$.

We write that the equilibrium position $\boldsymbol{x}_0$ of system [9.1] is Lyapunov stable if and only if

$$\forall\, \varepsilon \in \mathbb{R}^{+*},\ \exists\, \eta \in \mathbb{R}^{+*}, \forall\, \boldsymbol{x}_{10},\ \|\boldsymbol{x}_{10} - \boldsymbol{x}_0\| \leq \eta \implies$$
$$\forall\, n \in \mathbb{N},\ \|\mathcal{G}^n(\boldsymbol{x}_{10}) - \boldsymbol{x}_0\| \leq \varepsilon.$$

DEFINITION 9.5.– ***Asymptotic stability of periodic differential equations.*** An equilibrium position $\boldsymbol{x}_0$ of differential equation [9.1] is asymptotically stable if and only if the position is Lyapunov stable and there exists a neighborhood $\mathcal{V}$ of $\boldsymbol{x}_0$ such that for all initial condition $\boldsymbol{x}_{10}$ in $\mathcal{V}$ the solution $\mathcal{G}^n(\boldsymbol{x}_{10})$ of system [9.1] satisfies[1]

$$\lim_{n \to +\infty} \|\mathcal{G}^n(\boldsymbol{x}_{10}) - \boldsymbol{x}_0\| = 0.$$

DEFINITION 9.6.– Differential equation [9.1] is linear T-periodic if and only if the right-hand side of the equation can be written as

$$\boldsymbol{f}(\boldsymbol{x}, t) = f_1(t)\, \boldsymbol{x}$$

1 Let us note that for an autonomous differential equation, with the fixed point $\boldsymbol{x}_0$ of $\boldsymbol{f}(\boldsymbol{x})$, we can choose any period T for $\mathcal{G}$ since the mapping $f(\boldsymbol{x})$ does not depend on time. Since $\boldsymbol{x}_0$ is Lyapunov stable – respectively asymptotically stable – according to the definitions in Chapter 8, it is Lyapunov stable – respectively, asymptotically stable – according to Definitions 9.4 and 9.5 of this chapter.

where f_1 is a function from $\mathbb{R}$ to $\mathbb{R}$ of period T.

Index 1 will not be used, and we write

$$\frac{d\boldsymbol{x}}{dt} = f(t)\,\boldsymbol{x}. \qquad [9.4]$$

THEOREM 9.3.– Since differential equation [9.4] is linear, it is the same for mapping $\mathcal{G}$.

Indeed, as $\boldsymbol{x}_1$ is the solution of [9.4] associated with the initial condition $\boldsymbol{x}_{10}$ and as $\boldsymbol{x}_2$ is the solution of [9.4] associated with the initial condition $\boldsymbol{x}_{20}$, $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is the solution of [9.4] associated with the initial condition $\boldsymbol{x}_{10} + \boldsymbol{x}_{20}$. This result is still true for $t = T$ and therefore for $\mathcal{G}$.

9.3. Study of the planar case

9.3.1. *Generalities*

Let us consider the case of the periodic and linear differential equation [9.4]. The point $\boldsymbol{x}_0 = \mathbf{0}$ is a fixed point of $\mathcal{G}$.

DEFINITION 9.7.– A linear mapping $\mathcal{G}$ is Lyapunov stable if and only if its fixed point $\mathbf{0}$ is Lyapunov stable.

The definition is intrinsic because it is independent of the chosen basis to represent $\mathcal{G}$. Indeed, with $\mathcal{P}^{-1}\mathcal{G}\mathcal{P}$ denoting the matrix associated with $\mathcal{G}$ in the new basis associated with the change of basis represented by the matrix $\mathcal{P}$, we have $\|\mathcal{P}\| = \dfrac{\|\mathcal{P}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$. Let us demonstrate that for a given $\mathcal{P}$, there exist m and M strictly positive real numbers such that

$$m \leq \frac{\|\mathcal{P}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq M.$$

We prove that $m \leq \dfrac{\|\mathcal{P}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$ (for $\dfrac{\|\mathcal{P}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq M$, it is enough to replace $\mathcal{P}$ by $\mathcal{P}^{-1}$). On the contrary, there would exist $\boldsymbol{x}_1$ such that $\|\boldsymbol{x}_1\| = 1$ and $\dfrac{\|\mathcal{P}\boldsymbol{x}_1\|}{\|\boldsymbol{x}_1\|} = 0$; therefore, $\mathcal{P}\boldsymbol{x}_1 = \mathbf{0}$ and so $\mathcal{P}$ would not be invertible.

For $\boldsymbol{x}' = \mathcal{P}\boldsymbol{x}$, we have $\left\|\mathcal{P}^{-1}\mathcal{G}^n\mathcal{P}\boldsymbol{x}\right\| \leq \left\|\mathcal{P}^{-1}\right\| \|\mathcal{G}^n\boldsymbol{x}'\| \leq \dfrac{1}{m}\,\|\mathcal{G}^n\boldsymbol{x}'\|$. Since $\mathcal{G}$ is Lyapunov stable

$$\forall\,\varepsilon \in \mathbb{R}^{+*}, \exists\,\eta' \in \mathbb{R}^{+*}, \forall\,\boldsymbol{x}',\ \|\boldsymbol{x}'\| \leq \eta' \implies \forall\,n \in \mathbb{N},\ \|\mathcal{G}^n(\boldsymbol{x}')\| \leq \varepsilon\, m.$$

Furthermore,

$$\forall \eta' \in \mathbb{R}^{+*}, \exists \eta \in \mathbb{R}^{+*},\ \|\boldsymbol{x}\| \leq \eta \quad \Longrightarrow \quad \|\boldsymbol{x}'\| \leq \eta'.$$

Therefore,

$$\forall \varepsilon \in \mathbb{R}^{+*}, \exists \eta \in \mathbb{R}^{+*}, \forall \boldsymbol{x}, \quad \|\boldsymbol{x}\| \leq \eta \quad \Longrightarrow \quad \forall n,\ n \in \mathbb{N},\ \left\|\mathcal{P}^{-1}\mathcal{G}^n\mathcal{P}\boldsymbol{x}\right\| \leq \varepsilon$$

and $\mathcal{P}^{-1}\mathcal{G}\mathcal{P}$ is stable.

9.3.2. *Case where* $\det \mathcal{G} = 1$

In comparison with the Liouville theorem in mechanics, we say there is a *conservation of areas*.

THEOREM 9.4.– For the sake of simplicity, $\mathcal{G}$ also denotes the matrix of linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $\det \mathcal{G} = 1$:

– if $|\mathrm{Tr}(\mathcal{G})| < 2$, the mapping $\mathcal{G}$ is stable;

– if $|\mathrm{Tr}(\mathcal{G})| > 2$, the mapping $\mathcal{G}$ is unstable.

PROOF.– Let λ_1 and λ_2 be the eigenvalues of mapping $\mathcal{G}$. They satisfy the characteristic equation

$$\lambda^2 - \mathrm{Tr}(\mathcal{G})\,\lambda + 1 = 0, \text{ with discriminant } \Delta = [\mathrm{Tr}(\mathcal{G})]^2 - 4. \qquad [9.5]$$

First case: For $|\mathrm{Tr}(\mathcal{G})| < 2$, the roots λ_1 and λ_2 of [9.5] are complex conjugate; $\lambda_2 = \overline{\lambda_1}$ and $\lambda_1\lambda_2 = 1$, or equivalently $\lambda_1 = e^{i\alpha}$ and $\lambda_2 = e^{-i\alpha}$. The associated diagonal matrix representing $\mathcal{G}$ in the eigenvector basis is

$$\mathcal{G}_1 = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$$

and in a new basis of the two-dimensional complex vector space, we obtain the matrix representing $\mathcal{G}$,

$$\mathcal{G}_1 = \mathcal{P}^{-1}\mathcal{G}\,\mathcal{P}, \quad \text{with} \quad \mathcal{G} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

where $\mathcal{P}$ denotes the matrix of the change of coordinates. Furthermore, $\mathcal{G}$ is stable. Indeed,

$$\forall \varepsilon \in \mathbb{R}^{+*}, \exists \delta = \varepsilon,\ \forall \boldsymbol{x},\ \|\boldsymbol{x}\| \leq \varepsilon \implies \forall n \in \mathbb{N},\ \|\mathcal{G}^n(\boldsymbol{x})\| \leq \varepsilon,$$

because $\mathcal{G}^n$ is the rotation of angle $n\alpha$, which conserves distances to the fixed point $(0,0)$.

Second case: For $|\mathrm{Tr}(\mathcal{G})| > 2$, the roots λ_1 and λ_2 of [9.5] are real with, for example, $\lambda_1 > 1$ and $\lambda_2 = \lambda_1^{-1}$. For an appropriate change of coordinates' matrix $\mathcal{P}$, mapping $\mathcal{G}$ is represented by the matrix

$$\mathcal{G}_1 = \mathcal{P}^{-1}\mathcal{G}\mathcal{P} \text{ where } \mathcal{G}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \dfrac{1}{\lambda_1} \end{bmatrix}.$$

Furthermore, $\mathcal{G}_1$ is unstable. Indeed, the instability is written as

$$\begin{aligned} &\exists\, \varepsilon \in \mathbb{R}^{+*}, \forall\, \delta \in \mathbb{R}^{+*}\,, \exists\ \boldsymbol{x}_1,\ \|\boldsymbol{x}_1\| \leq \delta \\ &\text{and} \quad \exists\ n_1 \in \mathbb{N} \text{ such that } \|\mathcal{G}_1^{n_1}(\boldsymbol{x})\| > \varepsilon, \end{aligned} \tag{9.6}$$

with

$$\mathcal{G}_1^{n_1} = \begin{bmatrix} \lambda_1^{n_1} & 0 \\ 0 & \dfrac{1}{\lambda_1^{n_1}} \end{bmatrix}.$$

Let us choose $\varepsilon = 1$; for all δ belonging to $\mathbb{R}^{+*}$, let us choose $\boldsymbol{x}_1$ such that $\|\boldsymbol{x}_1\| = \dfrac{\delta}{2}$ associated with the eigenvalue $\lambda_1 > 1$, and n_1 such that $\lambda_1^{n_1}\dfrac{\delta}{2} \geq 1$ or $n_1 \geq \dfrac{\ln\left(\frac{2}{\delta}\right)}{\ln \lambda_1}$. Then, conditions [9.6] for instability is satisfied. ■

In the case $|\mathrm{Tr}(\mathcal{G})| = 2$, we cannot draw any general conclusions.

9.4. Strong stability in periodic Hamiltonian systems

We consider the case of the periodic differential equation [9.4] when $\det \mathcal{G} = 1$. The associated mechanical systems are said to be Hamiltonian.

DEFINITION 9.8.– Consider two T-periodic linear differential equations

$$\frac{d\boldsymbol{x}}{dt} = f_1(t)\,\boldsymbol{x} \quad \text{and} \quad \frac{d\boldsymbol{x}}{dt} = f_2(t)\,\boldsymbol{x}, \tag{9.7}$$

the maximum of distance between scalars $f_1(t)$ and $f_2(t)$ when t belongs to $[0,T]$ is called the distance between the two linear differential equations.

DEFINITION 9.9.– The trivial solution $\mathbf{0}$ of a T-periodic linear Hamiltonian system

$$\frac{d\boldsymbol{x}}{dt} = f(t)\,\boldsymbol{x} \qquad [9.8]$$

is strongly stable if and only if $\mathbf{0}$ is Lyapunov stable and if there exists a neighborhood of $f(t)$ such that, in this neighborhood, the Hamiltonian systems are also Lyapunov stable.

It follows that in the case of a plane – called the phase plane – we have the result:

THEOREM 9.5.– The trivial solution of differential equation [9.8] is strongly stable for $|\mathrm{Tr}(\mathcal{G})| < 2$.

Indeed, from the property of continuity of linear operators' trace, it follows that if $|\mathrm{Tr}(\mathcal{G})| < 2$, the mapping $\mathcal{G}'$ associated with a system close to $\dfrac{d\boldsymbol{x}}{dt} = f(t)\,\boldsymbol{x}$ is also such that $|\mathrm{Tr}(\mathcal{G}')| < 2$.

9.5. Study of the Mathieu equation. Parametric resonance

The Mathieu equation is the second-order scalar differential equation

$$\ddot{x} = -\omega^2\,(1 + \varepsilon\cos t)\;x, \qquad [9.9]$$

where $\omega \in \mathbb{R}^{+*}$ is called the *eigenfrequency* of [9.9] and $\varepsilon \in \mathbb{R}^{+*}$ is a *small parameter* ($\varepsilon \ll 1$). Let us note: the fact that ε is positive is of no importance. This is a particular case of the Hill equation corresponding to the motion differential equation of pendulum whose length varies periodically with a small amplitude of period 2π. The pendulum period is *close* to $\dfrac{2\pi}{\omega}$. The system of Hamilton equations associated with differential equation [9.9] is depicted by point (ω, ε) in the half-plane. Stable systems correspond to an open set in the half-plane such that $|\mathrm{Tr}(\mathcal{G})| < 2$; unstable systems correspond to $|\mathrm{Tr}(\mathcal{G})| > 2$; the limit of stability corresponds to $|\mathrm{Tr}(\mathcal{G})| = 2$.

THEOREM 9.6.– All points on the axis ω corresponding to $\varepsilon = 0$, except points $\left\{\omega = \dfrac{k}{2},\ k \in \mathbb{N}\right\}$, are associated with strong stable systems.

PROOF.– Determine the matrix of the mapping $\mathcal{G} = g_T$ for $\varepsilon = 0$ and $T = 2\pi$ in the basis of the phase plane associated with $x_1 = x$ and $x_2 = \dot{x}$. Equation [9.9] becomes

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\omega^2 x_1, \end{cases}$$

which has the general solution

$$\begin{cases} x = x_1 = c_1 \cos \omega t + c_2 \sin \omega t, \\ \\ \dot{x} = x_2 = -c_1 \omega \sin \omega t + c_2 \, \omega \cos \omega t. \end{cases}$$

– The solution corresponding for $t = 0$ to the initial conditions associated with the first-vector basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ defined by $x_1 = x = 1$ and $x_2 = \dot{x} = 0$ is

$$\begin{cases} x = \cos \omega t \\ \\ \dot{x} = -\omega \sin \omega t. \end{cases}$$

– The solution corresponding for $t = 0$ to the initial conditions associated with the second-vector basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ defined by $x_1 = x = 0$ and $x_2 = \dot{x} = 1$ is

$$\begin{cases} x = \dfrac{1}{\omega} \sin \omega t \\ \\ \dot{x} = \cos \omega t. \end{cases}$$

The linear mapping $\mathcal{G} = g_T$ associates the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with the vector

$$\begin{bmatrix} \cos \omega t \\ -\omega \sin \omega t \end{bmatrix}_{t=2\pi} = \begin{bmatrix} \cos 2\omega\pi \\ -\omega \sin 2\omega\pi \end{bmatrix},$$

and associates the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with the vector

$$\begin{bmatrix} \dfrac{1}{\omega} \sin \omega t \\ \cos \omega t \end{bmatrix}_{t=2\pi} = \begin{bmatrix} \dfrac{1}{\omega} \sin 2\omega\pi \\ \cos 2\omega\pi \end{bmatrix}.$$

Hence, $\mathcal{G} = g_T = \begin{bmatrix} \cos 2\omega\pi & \dfrac{1}{\omega} \sin 2\omega\pi \\ -\omega \sin 2\omega\pi & \cos 2\omega\pi \end{bmatrix}$ in the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We deduce $|\mathrm{Tr}(\mathcal{G})| = 2\,|\cos 2\omega\pi|$ and $|\mathrm{Tr}(\mathcal{G})| < 2$ for $\left\{ \omega \neq \dfrac{k}{2},\ k \in \mathbb{N} \right\}$. ■

It is possible to prove that the domain of instability of the Mathieu equation in the half-plane (ω, ε) keys the axis ω at points $\left\{ \omega = \dfrac{k}{2},\ k \in \mathbb{N} \right\}$. The corresponding

values of k are called the ***parametric resonance*** values. At these points on the half-plane (ω, ε), the linear systems may become unstable. This fact was experimentally observed for a *frictionless* swing, which becomes unstable if the user changes the distance of the mass center from its fixed pivot – even if it is a very small change – with a periodic variation; the variation is associated with the values of k. This *parametric resonance* manifests the strongest at $k = 1$, corresponding to the first point on the axis ω. In fact, in real material systems, friction intervenes and the equation to consider is no longer the Mathieu equation but an equation of the form

$$\ddot{x} + \chi\,\dot{x} + \omega^2\,(1 + \varepsilon\cos t)\;x = 0,$$

where χ is a small parameter of friction. As k becomes smaller, the domain of instability comes closer to the axis ω. Experimentally, the parametric resonance only manifests for $k = 1$, $k = 2$ and, much more rarely, for $k = 3$. Indeed, stability problems are difficult to study as it is rarely possible to make explicit $\mathcal{G} = g_T$. Most often, we determine the trace of $\mathcal{G}$ by a numerical integration of the motion equation over interval $[0, T]$. In the particular case where $f(t)$ is close to a constant, it is possible to use simple reasoning. This is the case we now consider.

9.6. A completely integrable case of the Hill equation

Determine the domain of stability of the dynamical system represented by the Hill equation:

$$\ddot{x} = -f^2(t)\,x, \qquad [9.10]$$

where

$$f(t) = \begin{cases} \omega + \varepsilon & \text{for } 0 < t \leq \pi \\ \omega - \varepsilon & \text{for } \pi < t \leq 2\pi \end{cases} \quad \text{and} \quad \forall\, t \in \mathbb{R}^{+*},\; f(t + 2\pi) = f(t)$$

with $\omega \in \mathbb{R}^{+*}$ and $\varepsilon \in \mathbb{R}^{+*}$ such that $\varepsilon \ll \inf(1, \omega)$.

Differential equation [9.10] can be written in matrix form $\dot{\boldsymbol{X}} = F(t)\,\boldsymbol{X}$, where $F(t)$ is linear:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f^2(t)\,x_1 \end{cases} \iff \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = F(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{with} \quad F(t) = \begin{bmatrix} 0 & 1 \\ -f^2(t) & 0 \end{bmatrix}.$$

The mapping $\mathcal{G}$ corresponding to the flow over period $T = 2\pi$ is linear. Let us write $g' = g_T \circ \left(g_{(T/2)}\right)^{-1}$. The mapping g' is linear because g_T and $g_{(T/2)}$ are linear. It is the same for g_t, associated with the flow at the instant t when $t = 0$ is the origin of time. With the coefficient $\omega_1 = \omega + \varepsilon$ being constant, differential equation [9.10]

is integrable on $]0,\pi]$; similarly, with the coefficient $\omega_2 = \omega - \varepsilon$ being constant, the differential equation is integrable on $]\pi, 2\pi]$. Let us determine $g_{(T/2)}$ and g' in the canonical basis of the phase plane

$$\begin{bmatrix}1\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\1\end{bmatrix} \text{ corresponding to } \begin{cases}x=1\\ \dot{x}=0\end{cases} \text{ and } \begin{cases}x=0\\ \dot{x}=1\end{cases}, \text{ respectively.}$$

Computation of $g_{(T/2)}$ on $[0,\pi]$

Differential equation [9.10] yields

$$x = b_1 \cos\omega_1 t + b_2 \sin\omega_1 t \quad \text{and} \quad \dot{x} = -b_1\,\omega_1 \sin\omega_1 t + b_2\,\omega_1 \cos\omega_1 t.$$

a)

$$\text{For } t=0, \quad \begin{cases}x=1\\ \dot{x}=0\end{cases} \quad \Longrightarrow \quad \begin{cases}b_1=1\\ b_2=0\end{cases},$$

hence,

$$\begin{cases}x=\cos\omega_1 t\\ \dot{x}=-\omega_1\sin\omega_1 t.\end{cases}$$

For $t=\pi$,

$$\begin{cases}x=\cos\omega_1\pi\\ \dot{x}=-\omega_1\sin\omega_1\pi.\end{cases}$$

b)

$$\text{For } \ t=0, \quad \begin{cases}x=0\\ \dot{x}=1\end{cases} \quad \Longrightarrow \quad \begin{cases}b_1=0\\ b_2=\dfrac{1}{\omega_1}\end{cases},$$

hence,

$$\begin{cases}x=\dfrac{1}{\omega_1}\sin\omega_1 t\\ \dot{x}=\cos\omega_1 t.\end{cases}$$

For $t=\pi$,

$$\begin{cases}x=\dfrac{1}{\omega_1}\sin\omega_1\pi\\ \dot{x}=\cos\omega_1\pi.\end{cases}$$

Finally,

$$g_{(T/2)} = \begin{bmatrix} \cos\omega_1\pi & \dfrac{1}{\omega_1}\sin\omega_1\pi \\ -\omega_1\sin\omega_1\pi & \cos\omega_1\pi \end{bmatrix}.$$

By writing $c_1 = \cos\omega_1\pi$ and $s_1 = \sin\omega_1\pi$, we obtain

$$g_{(T/2)} = \begin{bmatrix} c_1 & \dfrac{1}{\omega_1}s_1 \\ -\omega_1 s_1 & c_1 \end{bmatrix}.$$

Computation of g' on $[\pi, 2\pi]$

Differential equation [9.10] yields

$$x = d_1\cos\omega_2 t + d_2\sin\omega_2 t \quad \text{and} \quad \dot{x} = -d_1\,\omega_2\sin\omega_2 t + d_2\,\omega_2\cos\omega_2 t.$$

a)

$$\text{For } t = \pi, \quad \begin{cases} x = 1 \\ \dot{x} = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} d_1 = \cos\omega_2\pi \\ d_2 = \sin\omega_2\pi \end{cases},$$

hence,

$$\begin{cases} x = \cos\omega_2\pi\ \cos\omega_2 t + \sin\omega_2\pi\ \sin\omega_2 t \\ \dot{x} = -\,\omega_2\cos\omega_2\pi\ \sin\omega_2 t + \omega_2\sin\omega_2\pi\ \cos\omega_2 t. \end{cases}$$

For $t = 2\pi$,

$$\begin{cases} x = \cos\omega_2\pi \\ \dot{x} = -\,\omega_2\sin\omega_2\pi. \end{cases}$$

b)

$$\text{For } t = \pi, \quad \begin{cases} x = 0 \\ \dot{x} = 1 \end{cases} \quad \Longrightarrow \quad \begin{cases} d_1 = -\dfrac{1}{\omega_2}\sin\omega_2\pi \\ d_2 = \dfrac{1}{\omega_2}\cos\omega_2\pi \end{cases},$$

$$\begin{cases} x = -\dfrac{1}{\omega_2}\sin\omega_2\pi\ \cos\omega_2 t + \dfrac{1}{\omega_2}\cos\omega_2\pi\ \sin\omega_2 t \\ \dot{x} = \ \sin\omega_2\pi\ \sin\omega_2 t + \cos\omega_2\pi\ \cos\omega_2 t. \end{cases}$$

For $t = 2\pi$,

$$\begin{cases} x = \dfrac{1}{\omega_2} \sin \omega_2 \pi \\ \dot{x} = \cos \omega_2 \pi. \end{cases}$$

Therefore, g' can be written as

$$g' = \begin{bmatrix} c_2 & \dfrac{1}{\omega_2} s_2 \\ -\omega_2 s_2 & c_2 \end{bmatrix} \quad \text{with } c_2 = \cos \omega_2 \pi \text{ and } s_2 = \sin \omega_2 \pi,$$

and

$$g_T = g' \circ \left(g_{T/2}\right) = \begin{bmatrix} c_2 & \dfrac{1}{\omega_2} s_2 \\ -\omega_2 s_2 & c_2 \end{bmatrix} \begin{bmatrix} c_1 & \dfrac{1}{\omega_1} s_1 \\ -\omega_1 s_1 & c_1 \end{bmatrix}.$$

Finally, we obtain

$$\left|\mathrm{Tr}\left(g_T\right)\right| = \left|\mathrm{Tr}\, \mathcal{G}\right| = \left| 2c_1 c_2 - s_1 s_2 \left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} \right) \right|.$$

We compare $|\mathrm{Tr}\, \mathcal{G}|$ and 2, with $\mathrm{Tr}\, \mathcal{G}$ being an implicit function of ω and ε.

Study of $\mathrm{Tr}\, \mathcal{G}$

We have

$$\begin{cases} 2c_1 c_2 = 2 \cos \pi \left(\omega + \varepsilon\right) \cos \pi \left(\omega - \varepsilon\right) = \cos 2\pi\varepsilon + \cos 2\pi\omega, \\ 2s_1 s_2 = \cos 2\pi\varepsilon - \cos 2\pi\omega. \end{cases}$$

Furthermore,

$$\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} = \frac{\omega_1^2 + \omega_2^2}{\omega_1 \omega_2} = \frac{\left(\omega + \varepsilon\right)^2 + \left(\omega - \varepsilon\right)^2}{\omega^2 - \varepsilon^2} = 2\left(1 + \frac{2\varepsilon^2}{\omega^2}\right) + \mathcal{O}\left(\varepsilon^4\right).$$

Let us note

$$\Delta = \frac{2\varepsilon^2}{\omega^2} + \mathcal{O}\left(\varepsilon^4\right),$$

then,

$$\begin{aligned}\operatorname{Tr}\mathcal{G} &= \cos 2\pi\varepsilon + \cos 2\pi\omega - (1+\Delta)(\cos 2\pi\varepsilon - \cos 2\pi\omega)\\ &= -\Delta\cos 2\pi\varepsilon + (2+\Delta)\cos 2\pi\omega.\end{aligned}$$

The boundaries of stability correspond to

$$-\Delta\cos 2\pi\varepsilon + (2+\Delta)\cos 2\pi\omega = \pm 2.$$

Two cases are deduced:

First case:

$$\cos 2\pi\omega = \frac{2+\Delta\cos 2\pi\varepsilon}{2+\Delta}.$$

Since $\cos 2\pi\omega \simeq 1$ and ω is of the form $\omega = k+\alpha$, with $|\alpha| \ll 1$, $k \in \mathbb{N}$,

$$\cos 2\pi\omega = \cos 2\pi\alpha = 1 - 2\pi^2\alpha^2 + \mathcal{O}(\alpha^4).$$

Furthermore,

$$\cos 2\pi\omega = 1 - \frac{\Delta}{2+\Delta}(1-\cos 2\pi\varepsilon).$$

Therefore,

$$2\pi^2\alpha^2 + \mathcal{O}(\alpha^4) = \Delta\,\pi^2\varepsilon^2 + \mathcal{O}(\varepsilon^4) \text{ with } \omega - k = \pm\frac{\varepsilon^2}{\omega} + o(\varepsilon^2),$$

which implies

$$\omega = k \pm \frac{\varepsilon^2}{k} + o(\varepsilon^2). \qquad [9.11]$$

The boundary stability curves represented by equations [9.11] are approximately small arcs of parabolas. Indeed, for small ε, the domain of instability is very thin (thickness of ε^2 order).

Second case:

$$\cos 2\pi\omega = \frac{-2+\Delta\cos 2\pi\varepsilon}{2+\Delta}.$$

Since $\cos 2\pi\omega = -1 + o(\varepsilon)$ and ω is of the form $\omega = k + \dfrac{1}{2} + \alpha$ with $|\alpha| \ll 1$, $k \in \mathbb{N}$,

$$\cos 2\pi\omega = \cos(2\pi\alpha + \pi) = -1 + 2\pi^2\alpha^2 + o(\alpha^2),$$

or

$$\cos 2\pi\omega = -1 + \frac{\Delta}{2+\Delta}(1 + \cos 2\pi\varepsilon).$$

Therefore,

$$2\pi^2\alpha^2 + o(\alpha^2)\Delta \text{ with } \omega - k - \frac{1}{2} + o(\alpha^2) = \pm\frac{\varepsilon}{\pi\omega} + o(\varepsilon),$$

which implies

$$\omega = k + \frac{1}{2} \pm \frac{\varepsilon}{\pi\left(k + \dfrac{1}{2}\right)} + o(\varepsilon). \qquad [9.12]$$

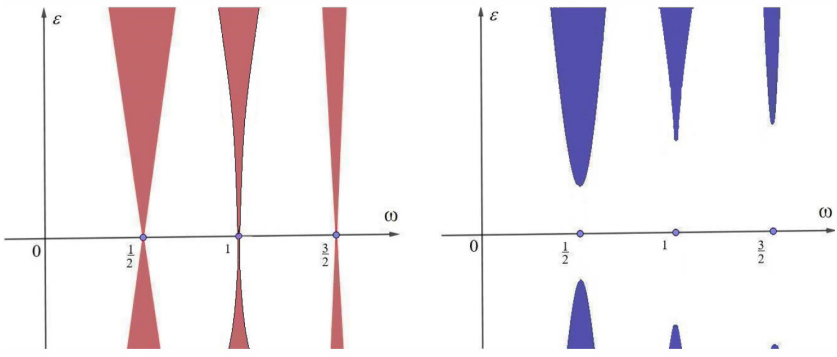


Figure 9.1. *The graph on the left-hand side corresponds to perfect case. The graph on the right-hand side corresponds to the case with friction. The colored zones represent the domains of instability due to small periodic variations of frequency ω. For a color version of this figure, see www.iste.co.uk/gouin/mechanics.zip*

The stability boundary curves represented by equations [9.12] are half-lines passing through the points $\omega = k + \dfrac{1}{2}$. The interior of colored zones whose thickness is of ε order are domains of instability: the graph on the left-hand side of Figure 9.1. In the case of friction, the arcs of curves are deformed and do not reach axis ω. For the domains of instability, we obtain (here, this has not been demonstrated) the interior of colored zones in the graph on the right-hand side of Figure 9.1.

10

Problems and Exercises

Exercise 1

At equilibrium, determine the plane curves of length L of a homogeneous flexible and inextensible wire connecting two given points A, B and subject to gravity acceleration.

Exercise 2

Prove that a necessary and sufficient condition for a curve to be traced on a sphere is that the radius of curvature R and the radius of torsion τ are related by the relation

$$\frac{d}{ds}\left(\tau\,\frac{d\mathrm{R}}{ds}\right)+\frac{\mathrm{R}}{\tau}=0.$$

Exercise 3

Study of the equilibrium interface of an incompressible liquid between two rectangular vertical flat plates. A knowledge of the concept of fluid capillarity is necessary for understanding this exercise. The interfacial section with an orthogonal plane at one extremity is represented by an unknown function $y=y(x)$. It is assumed that ℓ is very large with respect to $b-a$ (see the following figure).

1) Calculate the volume V of liquid in the parallelepipedic domain given in the figure.

2) Calculate the total potential energy due to gravity W_p in the parallelepipedic domain; W_{c_1}, the capillary energy in the form $\sigma_1\,S$, where σ_1 is a constant that represents the liquid–wall superficial tension and S is the area of the contact surface of liquid with walls along ℓ; and W_{c_2}, the capillary energy in the form $\sigma_2\,\Sigma$, where

σ_2 is a constant representing the liquid–air superficial tension and Σ is the area of the liquid interface along ℓ.

3) Find three relations relating x, y, σ_1 and σ_2.

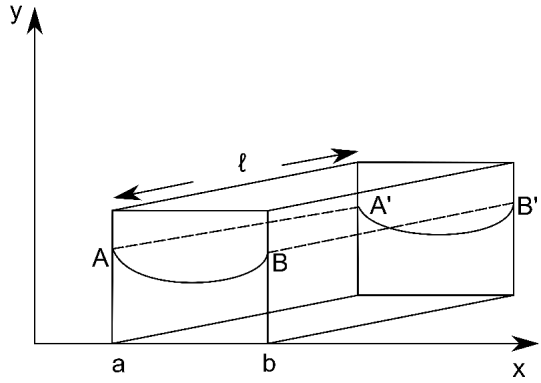
4) a) Prove that the connecting angles φ of the curved interface at points A and B are such that

$$\sin \varphi_A = -\sin \varphi_B = \frac{\sigma_1}{\sigma_2}.$$

b) Prove that curve $y = y(x)$ satisfies the Jurin equation written as

$$\rho\, g\, (y - y_0) = \frac{\sigma_2}{\mathrm{R}},$$

where ρ denotes the constant density of the liquid, g denotes the acceleration due to gravity, R denotes the local radius of curvature of curve $y = y(x)$ and y_0 is an integration constant.



Exercise 4

The notations of Exercise 3 are valid for the case of a liquid contained between two coaxial vertical circular cylinders made up of the same material with radii r_0 and r_1, where $r_0 < r_1$, and closed at its bottom by a horizontal disk.

1) Calculate the volume V of liquid in the domain between the two cylinders.

2) Calculate the total potential energy due to gravity W_p of the liquid domain; W_{c_1}, capillary energy of the form $\sigma_1\, S$, where σ_1 is a constant representing the liquid–wall superficial tension and S is the area of the contact surface of liquid with walls; and W_{c_2}, capillary energy of the form $\sigma_2\, \Sigma$, where σ_2 is a constant representing the liquid–air superficial tension and Σ is the interface area of liquid air (we said meniscus).

3) Prove that the meridian curve $z = z(r)$, with $r_0 \leq r \leq r_1$, of the meniscus satisfies the equation called Jurin's equation, where ρ denotes the constant density of the liquid, g denotes the acceleration due to gravity and R_1 and R_2 are the principal radii of curvature of the meridian curve.

4) Prove that the connecting angles φ of the meridian of the meniscus and the walls verify

$$\sin \varphi_{r_0} = -\sin \varphi_{r_1} = \frac{\sigma_1}{\sigma_2}.$$

Exercise 5

A surface (S) has equation $F(M) = 0$. Among all curves on (S) and connecting two given points A and B, we look for the curves whose length is extremal (they are called geodesics of the surface).

1) In a parametric representation, the curve is written as

$$M = f(u), \quad u \in [\alpha, \beta] \quad \text{with} \quad A = f(\alpha), \quad B = f(\beta)$$

At any point, we denote by $\boldsymbol{T}$ the unit vector of the oriented tangent to the curve, $\boldsymbol{N}$ its principal unit normal and R its radius of curvature.

Prove that

$$\frac{\boldsymbol{N}}{\mathrm{R}} + \lambda(u) \left(\frac{du}{ds} \right) \mathrm{grad} F = 0$$

and conclude that one characteristic property of the geodesics of the surface (S) is that the osculating plane of the geodesic is normal to (S).

2) Geodesics of the sphere:

a) prove that the geodesics of the sphere are parts of great circles,

b) obtain the result again by finding the extremum of the integral $L = \displaystyle\int_A^B ds$.

3) Study the geodesics of circular cylinders.

4) Study the geodesics of the cone of revolution:

a) by writing the equation of a cone in orthonormal axes as $r = az$, where (r, θ, z) is the cylindrical representation,

b) by writing the equation of the cone in orthonormal axes as $\theta = \theta(z)$.

5) Study of the geodesics of a surface expressed as a function of two parameters (u, v).

a) Prove that the first fundamental form of the surface is

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2.$$

Write ds^2 if coordinate lines are orthogonal.

b) Let $ds^2 = \dfrac{du^2 + dv^2}{u^2}$ be the first fundamental form of a surface (S), expressed as a function of parameters (u, v), where u is assumed to be positive. The point $M(u, v)$ of (S) corresponds to the point P whose coordinates in an auxiliary plane (Π) are (u, v). Determine the geodesics of the surface (S) and their images in the plane (Π).

Exercise 6

Let $F(M) = 0$ be the equation of surface (S) and let A and B be two given points on the surface. We recall that a necessary and sufficient condition for a curve on this surface, connecting A and B and written in a parametric representation as

$$M = f(u), \quad u \in [\alpha, \beta] \quad \text{with} \quad A = f(\alpha), \quad B = f(\beta),$$

to be of minimum length is

$$\frac{\boldsymbol{N}}{\mathrm{R}} + \lambda(u)\left(\frac{du}{ds}\right)\operatorname{grad} F = 0,$$

where $\boldsymbol{N}$ denotes the principal unit normal to the curve, R is its radius of curvature, s is the curvilinear abscissa and $\lambda(u)$ is an appropriate scalar field.

1) Since one of the characteristic properties of a helix is that its tangent vector makes a constant angle with a given fixed direction, prove that the geodesics on the cylinder with a circular base and the axis Oz are helices. Is it possible to forecast the result?

2) Prove that the geodesics on a surface of revolution are generally written as

$$r^2\,\frac{d\theta}{ds} = k$$

where k is constant. Obtain the result of 1) again.

Exercise 7

Let (C) be a plane curve connecting given points A and B. The plane orthonormal frame is Oxz. Consider the surface generated by (C) when Oxz rotates around Oz.

1) Determine (C) such that the area of the generated surface is minimal. Find the result by using,

a) the analytical calculus,

b) the geometrical calculus.

2) Determine (C) such that when the generated area is given, the volume resulting from the rotation of (C) is maximum.

Exercise 8

Consider $a = \int_{(C)} \boldsymbol{V}^\star d\boldsymbol{M}$, where $\boldsymbol{V}$ is a differentiable vector field in $\mathbb{R}^n$ and (C) is a differentiable curve connecting two given points A and B. The relation that gives the variation of a curvilinear integral is

$$\delta a = [\boldsymbol{V}^\star \delta \boldsymbol{M}]_A^B + \int_{\widehat{AB}} \delta \boldsymbol{V}^\star d\boldsymbol{M} - d\boldsymbol{V}^\star \delta \boldsymbol{M}.$$

In the Euclidean plane, determine the differential equation of curves (C) connecting A and B so that

$$a = \int_C n\, ds$$

is maximal, where n is a field that associates with each point M scalar $n = \phi(M)$.

Give in geometric form the differential equation that relates $\operatorname{grad} n$, the oriented unit tangent, the unit normal, and the algebraic radius of curvature of extremal curves.

Determine the equations of extremal curves when $\phi(M) = e^y$, where y is the ordinate of M in a system of orthonormal axes.

Exercise 9

Equilibrium shape of a wire on a surface of revolution.

A. Analytical method. An inextensible homogeneous wire (C) of linear mass ρ, connecting two given points A and B, is located without friction on the surface (S) of

revolution of a vertical axis. The cylindrical representation of the surface of revolution is $r = f(z)$, where the direction z is the vertical ascendant. The acceleration due to gravity is denoted by g. The curve (C) is defined by $\theta = \theta(z)$.

1) Prove that the length element of the wire is

$$ds = \sqrt{[1 + f'^2(z) + f^2(z)\,\theta'^2]}\,dz.$$

2) Write the differential equation satisfied by the wire on surface (S) as a function of z, θ', f, f'.

3) Instead of writing $\theta = \theta(z)$ for the representation of (C), we choose $z = z(\theta)$. Write the differential equation satisfied by the wire on surface (S) as a function of z, z', f, f'.

4) Study the equilibrium shape of a wire on a vertical circular cylinder.

B. Geometrical method.

1) The wire connecting A and B, which may be confounded, is located on a surface (S) (not necessarily the surface of revolution) of the equation $F(M) = 0$. Prove that the differential equation of curve (C) is

$$(z - h)\,\frac{d\boldsymbol{T}}{ds} - (\mathbf{1} - \boldsymbol{T}\,\boldsymbol{T}^{\star})\,\mathrm{grad}\,(z - h) - \mu\,\mathrm{grad}\,F = 0,$$

where h and μ are constants and $\boldsymbol{T}$ is the unit tangent vector to the curve (C).

2) Surface (S) is the surface of revolution around the axis Oz. A point M on the surface has coordinates (r, θ, z) in the moving system $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$ of the cylindrical representation of the axis Oz ($\boldsymbol{OM} = r\,\boldsymbol{u} + z\,\boldsymbol{k}$).

a) Write the components of $\boldsymbol{T}$ in $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$ frame. Deduce the matrix $\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star}$ and components of $\dfrac{d\boldsymbol{T}}{ds}$ in the basis $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$.

b) The coordinate lines of the surface are defined by $z = cte$ (called "parallels") and $\theta = cte$ (called "meridians").

i) Determine the components of unit vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ tangent to the coordinate lines in the basis $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$.

ii) Determine the components of the unit normal $\boldsymbol{n}$ to the surface (S) in the basis $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$.

3) From 2a), 2b) and 1), deduce the three scalar equations satisfied by the curve (C).

Integrate the simplest of the equations to obtain the result derived from method A, i.e. the equation

$$(z-h)\,r^2\,\frac{d\theta}{ds} = Cte.$$

Exercise 10

An inextensible homogeneous and flexible wire is located on the surface of a cone of revolution of the vertical downward axis Oz. The acceleration due to gravity is denoted by g. The cylindrical representation is (r, θ, z), and O denotes the vertex of the cone. The equation of the cone is

$$r = z\,\mathrm{tg}\,\alpha, \quad z > 0,$$

where α denotes the half-angle of the cone.

We will study two cases:

– case a: the wire connects given points (z_0, θ_0, r_0) and (z_1, θ_1, r_1) on the cone;

– case b: the wire is a closed curve rotating only once around the cone axis; the highest and lowest points of the wire are at levels z_0 and z_1, respectively.

1) By writing that the potential of the wire with a given length is minimum, obtain the differential equation that relates z and θ of the equilibrium shape. Prove that the equation can be written as

$$r\,(h-z)\cos i = k,$$

where h and k are constants and i is the angle between the tangent to the wire and the corresponding parallel at the point associated with (r, z, θ). This equation gives θ as an integral function of z. For case a, write the two conditions that determine the constants h and k.

2) For case b, prove that there exists an equilibrium position for which the wire is a horizontal circle. Assuming that there is another solution, calculate h and k as functions of z_0 and z_1 and write the conditions that determine z_0 and z_1. Discuss the existence of such a solution according to the values of angle α.

Exercise 11

Consider the half-plane of Oxy, where $x > 0$. We search for the extremals of the curvilinear integral $\displaystyle\int_{\widehat{AB}} \frac{ds}{x^m}$.

1) Prove that along an extremal, we have the relation:

$$x^m = a^m \sin\alpha,$$

where α is the angle made by the tangent of the integral curve with the x axis and a is constant. We choose $0 \leq \alpha \leq \pi$. What is the nature of a?

2) If $m \neq 0$, the parametric equations of the extremals are written as

$$x = a \sin^r \alpha, \quad y = y_0 + r \int_{\frac{\pi}{2}}^{\alpha} x \, d\alpha \quad \text{with } r = \frac{1}{m}.$$

What are the extremal curves obtained for $m = 0$?

3) Write the integral giving y for $m = 1$, $m = \frac{1}{2}$, $m = -1$, $m = -\frac{1}{2}$. What is the nature of the corresponding curves?

4) Prove that for the given m, the extremals are deduced from one of them by translation parallel to Oy and homothety of center O. Prove that each curve has an axis of symmetry.

5) Prove that if $m > 0$, concavity of the extremal curve is towards $x < 0$. Prove that there exists an extremal curve connecting any two given points A and B in the half-plane $x > 0$.

6) Prove that if $m < 0$, then no extremal or two extremal curves (exceptionally, a single extremal curve) connect any two given points A and B in the half-plane $x > 0$.

Exercise 12

Consider the motions of a point P moving in the half-plane $x > 0$ from a given point A at instant $t = 0$ to another given point B at instant $t = T > 0$.

1) Determine the motion such that

$$I = \int_0^T \sqrt{(a\,x)^2 - v^2}\, dt,$$

is extremal, where v is the velocity of P and a is a constant with the dimension of inverse time and chosen to be equal to 1. Such a motion is called the *extremal motion*.

To obtain this aim, it is possible to take t and y as unknown functions of x. Extremum of I is expressed by two first integrals. By eliminating y', we can express t as a function of x; we will write $x = \dfrac{a}{\cosh u}$ and y as a function of u.

2) Consider the extremal motions between points A and B such that A has coordinates $x = 1$ and $y = -1$, and B has coordinates $x = 1$ and $y = 1$.

a) What are the trajectories of these motions?

b) For which motions is the time lapse minimum? What is the corresponding trajectory and time lapse? (Note that the minimum time lapse is not zero).

3) Prove that the trajectory of an extremal motion with minimum time lapse is an extremal of $\displaystyle\int \frac{ds}{x}$. What are the corresponding trajectories?

Exercise 13

We search for the extremals $y = f(x)$ of

$$a = \int_{x_0}^{x_1} x\, y\, \sqrt{y'}\, dx.$$

1) Write the Euler equation satisfied by extremals; it is denoted by (1).

2) A parametric representation of the extremals is $x(t)\ y(t)$ and conjugate variables are p and q; write the corresponding Hamiltonian $H(p, q, x, y)$ and the Hamilton equations.

3) Note that the integral a is invariant by the transformation

$$\mathcal{T}_t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t\, x \\ \dfrac{y}{t} \end{bmatrix}.$$

Deduce the first integral $u = p\, x - q\, y$ denoted by (2) and write the first-order differential equation (3) in the form $F(x, y, y', u) = 0$ verified by the extremals.

4) By using H, verify whether u is a first integral. Prove that (3) is a consequence of (1).

5) Write the Jacobi equation of the extremals and find a solution in the form $S(x, y) = S_1(x) + S_2(y)$. Deduce the extremals in the form $y = f(x)$ denoted by (4).

6) From solution (4), prove that $v = \dfrac{p}{x^2}$ is another first integral. What is the corresponding invariance group? Can this invariance be directly obtained from the integral a? Is it possible to deduce other linear first integrals from u and v? Examine their invariance group.

7) In the plane, can two points of coordinates (x_0, y_0) and (x_1, y_1) always be connected by an extremal curve? Calculate the corresponding value of a. Is the extremum of a maximum or minimum value?

Exercise 14

A. The brachistochrone–analytical method

A point M of mass 1, without an initial velocity, describes without friction a curve connecting origin O and the given point A situated in a vertical plane. We search for curve (C) for which the time lapse is minimum. The curve is described in orthonormal axes by $y = y(x)$, where y is the coordinate corresponding to the vertical ascendant direction. The acceleration due to gravity is denoted by g.

1) After using the energy conservation theorem, prove that the time lapse to go from $O(0,0)$ to $A(a, y(a))$, $y(a) < 0$ is

$$\tau = \int_0^a \sqrt{\frac{1+y'^2}{-2gy}}\, dx = \int_0^a f(y, y')\, dx.$$

2) Prove that $y(x)$ verifies the partial differential equation

$$f - y' \frac{\partial f}{\partial y'} = c$$

where c is a constant.

3) Integrate this equation. Characterize the obtained curves.

B. The brachistochrone–geometrical method

1) Prove that

$$\tau = \int_O^A \frac{\boldsymbol{T}^\star}{v}\, d\boldsymbol{M},$$

where $\boldsymbol{T}$ is the unit tangent to the curve (C) and v is the velocity of the point M.

2) $\delta\tau$ being zero for any $\delta\boldsymbol{M}$, prove that

$$-\frac{\rho}{v}\, \boldsymbol{N} + (1 - \boldsymbol{T}\boldsymbol{T}^\star)\ \mathrm{grad}\ \left(\frac{1}{\mathrm{v}}\right) = 0,$$

where $\boldsymbol{N}$ is the principal unit normal to the curve (C) and ρ is the curvature at M. What does tensor $(\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^\star)$ mean?

Deduce equation

$$\mathrm{R}\, v\, \boldsymbol{N}^{\star}\mathrm{grad}\left(\frac{1}{v}\right) = 1,$$

where R denotes the radius of curvature of (C) at M.

3) From expression of v used in I, prove that $y = y(x)$ satisfies the second-order differential equation independent of x,

$$-2yy" = 1 + y'^2.$$

4) By writing $z(y) = y'$, get the result obtained in I.

Exercise 15

A point M with mass m moves without friction along a curve (C) which connects a point A with a vertical plane (Π).

The orthogonal projection of A on (Π) is denoted by H; we write $\boldsymbol{AH} = \ell\, \boldsymbol{j}_0$, where $\boldsymbol{j}_0$ is unit vector, and $\boldsymbol{i}_0 = \boldsymbol{j}_0 \wedge \boldsymbol{k}_0$, where $\boldsymbol{k}_0$ denotes the vertical ascendant unit vector.

The acceleration due to gravity is denoted by g. We write $\boldsymbol{AM}.\boldsymbol{k}_0 = z$. Let s be a curvilinear abscissa of (C) such that $\dfrac{ds}{dt} = v > 0$, and $\boldsymbol{T}$ be the unit vector tangent to (C) such that $\dfrac{\boldsymbol{dM}}{dt} = v\, \boldsymbol{T}$.

1) By using energy conservation E, where $E > 0$, prove that the time lapse τ to reach plane (Π) satisfies

$$\sqrt{\frac{2E}{m}}\, \tau = \int_{(C)} \boldsymbol{T}^{\star}\, \frac{\boldsymbol{dM}}{\sqrt{1 - hz}},$$

where constant h is a function of g and E.

2) Determine the conditions verified by (C) such that τ is minimal. Deduce that

$$\frac{\boldsymbol{T} \wedge \boldsymbol{k}_0}{\sqrt{1 - hz}} = \boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant vector.

3) It is assumed that $\boldsymbol{C} = c\,\boldsymbol{j}_0$ and $\varphi = (\boldsymbol{j}_0, \boldsymbol{T})$. From $\mathrm{R} = \dfrac{ds}{d\varphi}$, where R is the radius of curvature of (C), give the curve (C) for which time τ is minimal. What point in plane (Π) is the curve extremity? Give the result as a function of initial conditions.

Exercise 16

Consider the system of differential equations

$$\begin{cases} \dfrac{dx}{du} = 2x \\ \dfrac{dy}{du} = 3. \end{cases}$$

1) Prove the system has a unique solution denoted by $M = \mathcal{T}_u\ (M_0)$ for a given initial condition M_0.

2) Prove that mappings $\mathcal{T}_u$ form a Lie group.

3) Determine the infinitesimal displacement associated with $\mathcal{T}_u$.

4) Study the same questions for the system of differential equations

$$\begin{cases} \dfrac{dx}{du} = y \\ \dfrac{dy}{du} = x \end{cases}$$

Exercise 17

Let $\mathcal{A}$ be a C^1 mapping,

$$\boldsymbol{X} \in \mathbb{R}^n \longrightarrow \boldsymbol{Q} \in \mathbb{R}^p.$$

The linear tangent mapping at $\boldsymbol{X}$ is denoted by $\dfrac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}$. With the generating function $G(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ with action $a = \displaystyle\int_{t_0}^{t_1} G(\boldsymbol{Q}, \dot{\boldsymbol{Q}})\, dt$ is associated the generating function

$$\Gamma(\boldsymbol{X}, \dot{\boldsymbol{X}}) = G\left(\mathcal{A}(\boldsymbol{X}), \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\, \dot{\boldsymbol{X}}\right),$$

with action $b = \displaystyle\int_{t_0}^{t_1} \Gamma(\boldsymbol{X}, \dot{\boldsymbol{X}})\, dt$.

1) a) Express δb as a function of $\delta \boldsymbol{X}$ and $\delta \dot{\boldsymbol{X}}$; express δb as a function of $\delta \boldsymbol{Q}$ and $\delta \dot{\boldsymbol{Q}}$. Deduce $\dfrac{\partial \Gamma}{\partial \boldsymbol{X}}$ and $\dfrac{\partial \Gamma}{\partial \dot{\boldsymbol{X}}}$ as functions of $\dfrac{\partial G}{\partial \boldsymbol{Q}}$ and $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}$.

b) Let (C) be a curve defined by the mapping $t \in I \longrightarrow \boldsymbol{X}(t)$, where I is an interval of $\mathbb{R}$. Prove that (C) is an extremal curve of a if and only if $(\gamma) = \mathcal{A}(C)$ is an extremal curve of b.

c) If $\mathcal{A}$ is a diffeomorphism, prove that (γ) is an extremal curve if and only if $(C) = \mathcal{A}^{-1}(\gamma)$ is an extremal curve of a.

2) Consider a single-parameter group of diffeomorphisms $\{T_u,\ u \in \mathbb{R}\}$ such that $\boldsymbol{Q}_u = T_u(\boldsymbol{Q})$ and $\dot{\boldsymbol{Q}}_u = \dfrac{\partial \boldsymbol{Q}_u}{\partial \boldsymbol{Q}}\, \dot{\boldsymbol{Q}}$. It is assumed that for any u in an interval J, the generating function G satisfies $G(\boldsymbol{Q}_u, \dot{\boldsymbol{Q}}_u) = G(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$. Let $\boldsymbol{W}$ be the infinitesimal displacement associated with T_u. Prove that quantity $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}\, \boldsymbol{W}$ is constant over extremals of action a associated with G.

Exercise 18

A point M_1 with mass m_1 moves without friction on a horizontal plane (Π). An inextensible wire passing through a hole of (Π) connects M_1 with a point M_2 of mass m_2. We study the motions of the system when M_2 moves along the vertical upwards Oz, where g is the acceleration due to gravity. We denote $\overline{OM_2} = z$ and (r, θ) as the polar representation of M_1 in (Π).

1) Write the Lagrange equations of motion. Analyze the terms of these equations and obtain a relation of the form $\dot{r}^2 = f(r)$.

2) Discuss the motion. Find the relation that exists between the initial conditions when M_2 remains fixed.

Exercise 19

A mechanical system has two degrees of freedom. Its kinetic energy is

$$2T = \frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

There is no potential energy.

1) With t and x being two secondary variables, write two first integrals and deduce the motion of the system (i.e. calculate x and y as functions of t).

2) The conjugate variables of x and y are denoted by p and q, respectively. Calculate the Hamiltonian of the system. Write the Hamilton equations and by integration obtain the results of (1).

3) Write the Jacobi equation and obtain the motion.

4) Writing that $2T$ is invariant by the homothety $(x, y) \to (\alpha x, \alpha y)$, where α is a constant scalar, obtain a third first integral of motion and prove that we obtain the trajectories of the motion without integrating.

Exercise 20

Consider two circles of the axis Oz. We look for different methods to determine the surface of revolution supported by the two circles and whose area is minimum. The surface is given by its meridian curve. We use the semi-polar representation $(r,\, \theta,\, z)$.

1) Using the Euler method, find the meridian curve in the form $z = f(r)$: determine the differential equation (E) verified by the meridian curve and integrate it.

2) For the representation $z = z(\alpha)\,,\ r = r(\alpha)$ of the meridian curve, write the Hamilton and Jacobi equations and obtain the equation (E) again.

3) Prove that the meridian curves are extremals of an integral of the form $\displaystyle\int_{(C)} n\, ds$, where $n = g(r)$ has to be made explicit.

4) Obtain equation (E) by using Noether's theorem and give a geometric property of the center of curvature of meridian curves thanks to the variation of curvilinear integrals.

Exercise 21

A mechanical system has two degrees of freedom. Its position is identified by two parameters x and y, and its kinetic energy is

$$2T = y^2\, \dot{x}^2 + x^2\, \dot{y}^2.$$

There is no potential energy. We call p and q the conjugate variables of x and y, respectively.

1) What are the Lagrangian $L(x,\, y,\, \dot{x},\, \dot{y})$ and the Hamiltonian $H\,(x,\, y,\, p,\, q)$ of the system. Deduce the Lagrange equations (L), the Hamilton equations (H) and the Jacobi equation (J).

2) Prove that the Lagrangian is invariant by transformations T_α depending on a parameter α and defined by

$$T_\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\, e^{\alpha} \\ y\, e^{-\alpha} \end{bmatrix}.$$

From Noether's theorem, deduce a linear first integral in p and q (this first integral u is written in the form $u = ap + bq$, where a and b are functions of x and y).

By using either (L) equations or (H) equations, prove that the dynamical variable u is a first integral.

3) Verify whether the dynamical variable

$$v = \frac{p}{y}\, \cos\left(\mathrm{Log}\, \frac{x}{y}\right) + \frac{q}{x}\, \sin\left(\mathrm{Log}\, \frac{x}{y}\right)$$

is another first integral. Deduce a third first integral w.

4) Using these three first integrals and without carrying out any integration, find the equations for trajectories.

5) Find a complete integral of the Jacobi equation when the generating function is written in the form

$$S(x,\, y,\, t,\, E,\, c) = -Et + f(\xi) + g(\eta)$$

with $\xi = xy$, $\eta = \dfrac{x}{y}$, where f and g are two functions to be determined. The complete integral depends on a parameter c, which is chosen as the value of the first integral u.

Deduce the equations of motion. Obtain the equations for the trajectory again.

Exercise 22

A mechanical system has two degrees of freedom. Its position is identified by two positive parameters x and y. Its kinetic energy is

$$2\, T = \frac{y}{x}\, \dot{x}^2 + \frac{x}{y}\, \dot{y}^2.$$

There is no potential energy. We call p, q and h the conjugate variables of x, y and t, respectively.

1) What is the Hamiltonian $H(x, y, p, q)$ of the mechanical system? Write the Lagrange equations (L), the Hamilton equations (H) and the Jacobi equations (J).

2) Write and justify the relation existing between x, y, p, q, expressing the first energy integral. The value of the energy is denoted by E.

3) Prove that the Hamiltonian action is invariant by the group of transformations τ_α depending on a parameter α and defined by

$$T_\alpha \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} x\, e^{\alpha} \\ y\, e^{-\alpha} \\ t \end{bmatrix}.$$

From Noether's theorem, deduce a linear first integral (1) in p and q (this first integral is written as $u = a\,p + b\,q$, a with b being the functions of x and y).

Verify whether this dynamical variable is a first integral by using either (L) equations or (H) equations.

4) Consider the group of transformations

$$T_\lambda \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} \lambda\, x \\ \lambda\, y \\ \lambda^n\, t \end{bmatrix}.$$

For what value of constant n is the group also an invariance group? Find another first integral, we denote by (2). Verify whether this dynamical variable is a first integral by using either (L) equations or (H) equations. Prove that there exists an instant t_0 where the expression $p\,x + q\,y$ is zero. The value of $p\,x$ at this instant t_0 is denoted by c. Express the values of first integrals (1) and (2) as functions of E, t_0 and c.

5) From the three first integrals and without integration, deduce a relation (3) between x, y, t and E, t_0, c.

6) We denote $\theta = \dfrac{1}{2}\,\mathrm{Log}\,\dfrac{x}{y}$; using (H) equations and first integrals of t, E, t_0 and c, calculate the quantity $\dfrac{d\theta}{dt}$. Deduce t as a function of θ (this relation will be denoted by (4)).

7) From (3) and (4), deduce the equation of trajectories relating x and y (this equation will be denoted by (5)).

8) By taking a generating function of the form

$$S(x, y, t, E, C) = -Et + f(\xi) + g(\eta),$$

find a complete integral of the Jacobi equation, with $\xi = xy$ and $\eta = \dfrac{x}{y}$, where f and g are functions to be determined; this integral depends on a parameter c. Deduce the equations of motion and relations (3) and (5).

Exercise 23

A mechanical system has two degrees of freedom. Its position is defined by parameters ρ and θ, and its kinetic energy is

$$2T = \rho^2\,[\dot{\rho}^2 + \rho^2\,\dot{\theta}^2].$$

There is no potential energy. We call p and q the conjugate variables of ρ and θ, respectively.

1) What is the Hamiltonian $H(\rho, \theta, p, q)$? Write the Lagrange equations and the Hamilton equations. Write the Jacobi equation for which a complete integral will be given and deduce the equations of motion.

In an auxiliary plane, the parameters ρ and θ denote the polar representation of the point M; verify the trajectories of M are conics and give the law of motion on the trajectories.

2) From the invariance by rotation of the Lagrangian around the origin in the polar representation, deduce a first integral (this first integral may also be directly obtained using Lagrange or Hamilton equations).

3) Prove that the dynamical variable $u = \dfrac{1}{\rho^2}\,(\rho\,p\,\cos 2\theta - q\,\sin 2\theta)$ is also another first integral.

4) From the two first integrals, deduce a third first integral independent of the previous ones.

5) By eliminating p and q between the three first integrals, obtain the trajectories of M again.

6) By using the Maupertuis principle, obtain the trajectories of M again.

Exercise 24

A mechanical system has two degrees of freedom. Its position is given by parameters x and y, which are strictly positive. The kinetic energy is

$$2T = (x + y)\left(\frac{\dot{x}^2}{x} + \frac{\dot{y}^2}{y}\right),$$

the potential independent of time of the applied forces to the mechanical system is denoted by $W(x, y)$. We call p and q the conjugate variables of x and y.

1) What is the Hamiltonian $H(x, y, p, q)$?

2) What is the most general form the potential $W(x, y)$ must have to obtain a separation of variables? In other words, the Jacobi partial derivatives will have a complete integral of the generating function of the form

$$S = \varphi(t) + f(x) + g(y)$$

3) Now, we assume $W(x, y) = 0$. What is the Hamiltonian? Write the Hamilton equations.

4) Using the Jacobi method, write the equations of motion. The equation expressing the time law is denoted by (I), the equation giving the trajectories is denoted by (II).

5) Prove that dynamical variable $\alpha = x(p^2 - 2H)$ is a first integral.

6) Find the function $b(x, y)$ of the two variables x and y, satisfying $b(1, 1) = 1$ and such that $w = (p - q)\, b(x, y)$ is a first integral. Deduce that $v = \dfrac{\alpha}{w}$ is another first integral.

7) As the Poisson bracket of two first integrals is a first integral, deduce the first integral $[v, w]$. We denote $u = \dfrac{px - qy}{x + y}$. Prove that u is a first integral. Calculate $u^2 + v^2$ as a function of H and, from u, deduce the value of $[u, v]$.

8) By eliminating p and q between the three first integrals u, v, w, obtain the expression (II) for the trajectories.

9) Prove that transformation

$$T_\lambda \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} \lambda\, x \\ \lambda\, y \\ \lambda^n\, t \end{bmatrix}$$

keeps invariant the Hamilton action for a value of n to be evaluated. Deduce a first integral and obtain (I) again.

10) Let G^1 be the invariance group corresponding to the first integral u and which is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = T_X^1 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

T_X^1 transformation of the group G^1.

By integrating the differential equation associated with G^1, prove that if X is the parameter of the group, we have

$$\begin{cases} x - y = x_0 - y_0 + X, \\ x\,y = x_0\,y_0. \end{cases}$$

11) Same question for the first integral v, with $T_Y^2 \in G^2$, G^2 being the invariance group corresponding to v and Y, the parameter. Prove that

$$\begin{cases} x - y = x_0 - y_0, \\ 2\sqrt{x\,y} = 2\sqrt{x_0\,y_0} + Y \end{cases}$$

12) Let us consider the change of variables $(X, Y) \longrightarrow (x, y)$

and $\begin{bmatrix} x \\ y \end{bmatrix} = T_X^1 \circ T_Y^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Let P and Q be the conjugate variables of X and Y, respectively. Calculate the new Hamiltonian associated with X, Y.

Exercise 25

Consider a point M with mass m moving on a surface of revolution (S) of the axis $O\boldsymbol{k}_0$, where O is the origin. The acceleration due to gravity is denoted by g. The constraints are perfect and independent of time, and the point M is not subject to other forces. We choose a cylindrical system of representation such that $\boldsymbol{OM} = r\,\boldsymbol{e}_r + z\,\boldsymbol{k}_0$. The angle between moving vector $\boldsymbol{e}_r$ and a fixed vector $\boldsymbol{i}_0$ orthogonal to $\boldsymbol{k}_0$ is denoted by θ.

1) Calculate the kinetic energy and the potential energy of the point M.

2) Write the differential equations of motion and deduce first integrals by using:

a) the Lagrange method,

b) the Hamilton method,

c) Noether's method.

3) Use the method of re-injecting partial result and what conclusion can you deduce?

Exercise 26

A point M with unit mass is attracted by a fixed center O by a force deriving from the potential $W = -\dfrac{k^2}{2\,r^4}$, where k is a given constant and r is the distance OM. As the trajectory is planar and the motion has central acceleration, we choose the polar representation (r, θ) to determine the position of M. The total energy of the material point is denoted by E and C is the constant of areas. The conjugate variables of r and θ are denoted by p_r and p_θ.

1) Calculate the Hamiltonian and write the Hamilton equations.

2) Deduce two first integrals.

Write the equation of the trajectory as $\theta - \theta_0 = \displaystyle\int_{r_0}^{r} F(r)\,dr$, $F(r)$ being given as a function of E, C and k.

Write the equation for the time law as $t - t_0 = \displaystyle\int_{r_0}^{r} G(r)\,dr$, $G(r)$ being given as a function of E, C and k. We do not integrate these two equations.

3) Study of the specific case of motion when the total energy E is zero.

a) Prove that $F(r) = \dfrac{C}{\sqrt{k^2 - C^2\,r^2}}$. Deduce the trajectories and identify the nature of the curves.

b) Prove that $G(r) = \dfrac{1}{\sqrt{\dfrac{k^2}{r^4} - \dfrac{C^2}{r^2}}}$. Deduce the time law.

4) Using the Maupertuis principle, define the trajectories as extremals of an integral of the form $\displaystyle\int_{s_0}^{s_1} n\,ds$. Express n as a function of E and r. Obtain the trajectories when $E = 0$ (we study the extremals of $a = \displaystyle\int_{\theta_0}^{\theta_1} \frac{\sqrt{r'^2 + r^2}}{r^2}\,d\theta$).

5) Write the Lagrange equations of motion and deduce that it is possible to have motions where r remains constant. What are the corresponding initial conditions? What is the relation between E and C?

Exercise 27

Consider a system (S) with two degrees of freedom. The position parameters are denoted by x and y and the associated conjugate variables are denoted by p and q, respectively.

1) System (S) is invariant by the group of transformations $\mathcal{T}_\alpha$ defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{T}_\alpha \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

such that

$$x = \frac{x_0 + \operatorname{tg}\alpha}{1 - x_0 \operatorname{tg}\alpha}, \quad y = y_0 \cos\alpha - x_0 \sin\alpha.$$

a) Verify whether the transformations form a Lie group.

b) Calculate the infinitesimal displacement $\boldsymbol{W}_1(x, y)$ of the group.

c) Give the first integral we can deduce from the group.

2) Let $\boldsymbol{W}_2(x, y) = \begin{bmatrix} \dfrac{1}{y} \\ 0 \end{bmatrix}$ be a second infinitesimal displacement.

a) Express the associated Lie group.

b) Give the second first integral deduced from this group.

3) By using Poisson's brackets, deduce a third first integral. Prove that this first integral expresses the invariance relative to a third group after having indicated the corresponding infinitesimal displacement.

4) Then, deduce trajectories of the motions of (S).

Exercise 28

A point of unit mass whose position is defined by the spherical representation (r, θ, φ) is subject to a force deriving from the potential W such that

$$W = f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2 \sin^2\theta},$$

where f, g and h are real regular functions of the real variable.

1) Form the Jacobi partial differential equation and give the solutions in the form of integrals. Deduce in the integral form the system (E) of the three equations of motion (these integrals give time law and trajectories).

2) Obtain system (E) again by using the Lagrange equations and the Hamilton equations.

3) When f, g and h are identically zero, calculate (E) and verify that the motion is what we expected.

Exercise 29

A system has three degrees of freedom. Its position is given by the dimensionless parameters x, y and z. Its kinetic energy T and potential energy W are given, respectively, by

$$2\,T = A\left(2\,\dot{x}^2 + \dot{y}^2\cos^2 x + \dot{z}^2\sin^2 x\right), \quad W = \frac{A\,\omega^2}{4\sin^2 2x}.$$

where A is a constant with dimension of moment of inertia and ω is a constant with dimension of angular velocity.

1) Find the motion of the system when the initial conditions are

$$x_0 = \frac{\pi}{4},\ \dot{x}_0 = \alpha\,\omega,\ \dot{y}_0 = \beta\,\omega,\ \dot{z}_0 = \gamma\,\omega,$$

where α, β, γ are three real constants. Write the equations of motion and verify it is possible to express $\dot{y}$ and $\dot{z}$ as functions of $u = \cos 2x$ and obtain u as a function of t.

2) Prove that x lies between 0 and $\dfrac{\pi}{2}$. For what values of α, β, γ will we have a motion for which x remains constant?

3) The system is subject to the perfect constraint

$$\dot{y} + \dot{z} = \varphi(x),$$

where $\varphi(x)$ is a continuous function of x. Prove that the equations of motion allow us to calculate $\dot{y}$ and $\dot{z}$ as a function of x. Then, obtain a differential equation of x. Express x, y, z as functions of time when $\varphi(x) = 0$.

Exercise 30

The change of variables $\begin{bmatrix} p \\ q \end{bmatrix} \longrightarrow \begin{bmatrix} r \\ s \end{bmatrix}$ is defined by

$$r = \alpha\,p + \beta\,q \qquad s = \beta\,p + \alpha\,q.$$

What relation must verify α and β for a canonical change of variables? Deduce the generating function $F(q, s)$.

Exercise 31

Consider a mechanical system with one degree of freedom whose phase is defined by the conjugate variables q and p and the change of variables

$$\begin{cases} s = \text{Ln}\left(\dfrac{\sin p}{q}\right) \\ \\ r = q \,\text{cotg}\, p \end{cases}.$$

1) Prove that the change of variables is canonical.

2) Calculate the generating function $F(s,p)$ of the change of variables when s and p are independent.

3) $G(q,r)$ is the generating function when q and r are independent. What is the relation between F and G? Deduce the function G.

Exercise 32

Consider a mechanical system with one degree of freedom whose phase is defined by the conjugate variables q and p and the change of variables

$$\begin{cases} s = p^{\alpha} \sin\, \beta\, q \\ \\ r = p^{\alpha} \cos\, \beta\, q \end{cases}$$

1) For which α and β values are the change of variable canonical?

2) Calculate the different generating functions $S(q,s)$, $F(q,r)$, $G(p,r)$, $H(p,s)$.

Exercise 33

Consider the canonical change of variables

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} s \\ r \end{pmatrix}$$

associated with generating function $S = -\dfrac{s^2}{2}\,\text{cotg}\, 2q$. What is the canonical change of variables?

Exercise 34

Consider the change of variables

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} s \\ r \end{pmatrix}$$

with

$$\begin{cases} r = 2\,(1 + \sqrt{q}\,\cos p)\,\sqrt{q}\,\sin p \\ \\ s = \mathrm{Ln}\,(1 + \sqrt{q}\,\cos p) \end{cases}.$$

1) Prove that the change of variables is canonical.

2) Calculate the generating functions $F(p, s)$ and $S(q, s)$ of the canonical changes of variables.

Exercise 35

Consider the planar motion of a point M of unit mass, attracted by two fixed points with forces in inverse-square distance. The Cartesian coordinates of the point M are given by (x, y) for the orthonormal axes with origin O, the midpoint of the segment that connects the two fixed points on the axis Ox. We denote by q_1 and q_2 the elliptical coordinates of the point M defined by

$$\begin{cases} x = \cos q_2\ \mathrm{ch}\, q_1 \\ \\ y = \sin q_2\ \mathrm{sh}\, q_1 \end{cases}.$$

1) Prove that the coordinate lines are ellipses and hyperbolas, respectively, whose foci F and F' are to be determined.

2) Calculate the kinetic energy of the point M in the form $T(q_1, q_2, \dot{q}_1, \dot{q}_2)$ and the potential energy in the form $W(q_1, q_2)$.

3) Write the Jacobi partial differential equation and, by using a simple integration, give the generating function S. Deduce the time law and equation of the trajectory of the point M.

4) If $H = E$, find a canonical change of variables associated with a generating function $F(\boldsymbol{Q}, \boldsymbol{R})$ such that the new Hamiltonian is of the form $K = K(\boldsymbol{R})$ (we will obtain a system decomposable in the sense of Liouville). Write the new Hamilton equations, the Jacobi equation and the equations of motion.

5) We must recall what a multi-periodic, dynamically decomposable system in the sense of Liouville is. Define the action variables, the angular variables and the frequencies with respect to q_1 and q_2.

Exercise 36

This exercise relates to the motion of planets using action variables and angular variables.

A planet P of mass m reduced to its center is attracted by another planet O assumed to be fixed and also reduced to its center, with a force in inverse-square distance (the force value is $-\dfrac{K}{r^2}$).

A. The plane of the orbit of P is assumed to be known, and in this plane, the coordinates of P are in polar representation (r, θ).

1) Calculate the kinetic energy, the potential energy and the Hamiltonian of P.

2) Write the Jacobi equation. Deduce the law of areas, the time law and the trajectory equation of P.

3) When the trajectory is closed ($r_1 \leq r \leq r_2$), the motion is periodic. Write the action variables J_r and J_θ in the integral form. Develop J_θ.

4) Prove the calculus of J_r is the same as the calculus of

$$\int_{r_1}^{r_2} \sqrt{E(r-r_1)(r-r_2)}\,\frac{dr}{r},$$

we denote by $\displaystyle\int_{r_1}^{r_2} f(r)\,dr$. What must the sign of E be?

5) The last integral can be calculated using simple calculus, but it is quicker to use the residue method (after having determined zeros and the pole of $f(r)$. Choose an appropriate path and calculate the residue for the pole). Prove that

$$J_r = -2\pi\sqrt{2m}\,\alpha + \frac{\pi K}{\sqrt{-E}}\sqrt{2m}.$$

Deduce the value of E as a function of $J_r + J_\theta$ (verify its sign).

6) By using the angular variables, calculate the frequencies of the motion and deduce the common period.

B. We are now in the three-dimensional space. Although we do not ignore the fact that the trajectory is planar, we use the spherical representation (r, φ, θ).

Study all questions of (A) again and prove that

$$E = -\frac{2m\,\pi^2\,K^2}{(J_r + J_\varphi + J_\theta)^2}.$$

Prove the three frequencies are equal.

Exercise 37

Consider the differential system

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_2 \\ \dfrac{dx_2}{dt} = x_1 + x_2 \end{cases}$$

1) Calculate its solutions. Determine the equilibrium position. What is the character of stability of the equilibrium position? What can you say about the transformation g^t associated with the differential system?

2) Consider the square of the phase plane: $|x_i| \leq 1$ where $i \in \{1, 2\}$. Determine its image at the instant $t = 1$. What is the geometrical nature of this image?

Exercise 38

1) Study the character of the stability of the equilibrium position of the differential equation

$$\ddot{x} + \sin\, x = 0.$$

Is the volume conserved in the phase space? Study the possible asymptotic stability.

2) Same question for

$$\ddot{x} + x^3 = 0.$$

Exercise 39

Consider the differential system

$$\begin{cases} \dfrac{dx}{dt} = -\sin y \\ \dfrac{dy}{dt} = \sin x + \sin y \end{cases}$$

where x and y are two real functions defined over $[0, 2\pi[$.

1) Determine equilibrium points of the differential system.

2) Then, write the matrix that represents the tangent linear mapping of the motion at these points.

3) Study the stability of equilibrium points.

Exercise 40

Frame $R_0 = (O, \boldsymbol{i}_0, \boldsymbol{j}_0, \boldsymbol{k}_0)$ is an orthonormal coordinate system, where $\boldsymbol{k}_0$ represents the direction of the ascendant vertical. A point M of unit mass, whose coordinates are (x, y, z), moves without friction on a surface $z = f(x, y)$. The point M is subjected to the acceleration due to gravity g.

1) Write the kinetic energy and the potential energy of the point M as functions of $x, y, f(x, y)$ and their time derivatives.

2) What conditions must satisfy f such that O is in stable equilibrium position?

3) We write

$$p(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0), \quad q(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$r(x_0, y_0) = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad s(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \quad t(x_0, y_0) = \frac{\partial^2 f}{\partial y^2}(x_0, y_0).$$

We recall that the principal radii of curvature of a surface at the point (x_0, y_0) are R_1 and R_2 such that

$$\frac{1}{2}\left(\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right) = \frac{(1+p^2)t + (1+q^2)r - 2pqs}{2(1+p^2+q^2)^{3/2}}$$

$$\frac{1}{\mathrm{R}_1 \mathrm{R}_2} = \frac{rt - s^2}{(1+p^2+q^2)^2}.$$

Calculate the eigenfrequencies of the system as functions of R_1 and R_2. Deduce the most general representation of the small motions near equilibrium position O.

Exercise 41

1) Write the kinetic energy, potential energy and Lagrange equations of a spherical pendulum consisting of a material point of mass m, subjected to gravity forces and in frictionless contact with a sphere of center O and radius R. Consider two parameters denoted by θ and ψ corresponding to the Euler angles (θ is the angle of the direction OM with the ascending vertical axis $O\boldsymbol{k}$ and ψ is the angle of the projection of OM with the axis $O\boldsymbol{i}$ of the horizontal plane).

2) We enforce the constraint $\dot{\psi} = \omega$, where ω is constant. Give the associated equation of motion.

3) What is the corresponding equilibrium position? From the linearized equation, study the small motions near this equilibrium position. What is the period of small motions?

Exercise 42

A double pendulum consists of two material points M_1 and M_2 which have the same mass m. The point M_1 is connected to a fixed point O with a wire of length ℓ; the point M_2 is connected to the point M_1 by another wire of the same length. The angles that OM_1 and M_1M_2 make with the vertical descending axis Ox are denoted by u and v, respectively.

1) Calculate the kinetic energy of the system.

2) The acceleration due to gravity is denoted by g. Calculate the potential energy of the system. For what values of u and v can we have a stable equilibrium position? The values associated with this position are denoted by u_0 and v_0.

3) We plan to study the small motions near the stable equilibrium position. We write $u = u_0 + \alpha$ and $v = v_0 + \beta$.

a) Write the linearized kinetic energy T_ℓ and the linearized potential energy W_ℓ.

b) Prove that the associated equations of motion form a second-order differential system with constant coefficients.

c) Find the small motions in the form

$$\alpha = A\cos\,\omega\,(t-\tau), \quad \beta = B\cos\,\omega\,(t-\tau).$$

Determine the two eigenvibrations ω_1 and ω_2 and obtain the most general motion of the system.

Exercise 43

This exercise is a study of the longitudinal vibrations of a linear triatomic molecule.

At equilibrium, a molecule consists of two atoms of mass m situated at A and C symmetrically located on both sides of an atom of mass M situated at B. The three atoms are aligned and at equilibrium, the distances AB and BC are equal to b. The intra-atomic potential is symbolized by the action of two springs that symmetrically connect the three atoms with the same spring constant k.

1) Let x_{oi} $(i = 1, 2, 3)$ be the respective equilibrium positions of the three atoms. Using coordinates $\eta_i = x_i - x_{oi}$ with respect to the equilibrium positions, write the kinetic energy and the potential energy of the molecule.

2) Deduce the eigenfrequencies ω of small oscillations. What does the value $\omega = 0$ mean?

3) It is assumed that the center of mass of the molecule remains fixed at origin O.

a) What is the relation between m, M, x_1, x_2, x_3?

b) Study the eigenfrequencies of the molecule and conclude.

4) We consider the frame of eigen-directions associated with the eigenfrequencies α, β, γ. In this frame, the kinetic energy and the potential energy are

$$T = \frac{1}{2}\, m \sum_i \dot{Q}_i^2, \quad W = \frac{1}{2}\, m \sum_i w_i^2\, Q_i^2.$$

Study the normal modes corresponding to the different eigenfrequencies. How do we obtain longitudinal vibrations without rigid translation of the molecule?

Exercise 44

A mechanical system subject to gravity $g = 1$ consists of two identical simple pendulums of lengths $\ell_1 = \ell_2 = 1$ and masses $m_1 = m_2 = 1$. The pendulums are connected by a spring without mass and whose length, when the spring is unstressed, is equal to the distance $\ell = 1$ between their two points of attachment on a horizontal axis. The angles made by the two pendulums with respect to the descendant vertical axis are denoted by θ_1 and θ_2, respectively.

1) Calculate the kinetic energy of the system.

2) Let k be the spring coefficient. Calculate the potential energy of the system. Determine the equilibrium position. Calculate the linearized kinetic energy and potential energy near the equilibrium position.

3) Determine the two eigenfrequencies of the system in small motion near the stable equilibrium position and give the most general linearized motion of the system.

4) To which pendulum motion does each eigenfrequency correspond?

5) It is assumed that $k \ll 1$ and at initial instant $t = 0$, $\theta_1 = \theta_2 = 0$ and $\dot{\theta}_1 = v$, $\dot{\theta}_2 = 0$. Prove that after a time $\mathcal{T}$, the first pendulum will have no velocity, all the energy being concentrated in the second pendulum.

Exercise 45

On the horizontal axis $\boldsymbol{Oi}_0$, an unstressed spring of length $4\,a$ and spring coefficient k is fixed at both ends. Three material points A, B, C of masses m, $\frac{4}{3}\,m$ and m, respectively, are attached to the spring at equal distance a and distance a between the spring ends. The oscillation of the spring occurs on the axis $\boldsymbol{Oi}_0$ (then

gravity is not taken into account). The system produces small oscillations. We wish to study the motion of the mechanical system.

1) Calculate the kinetic energy of the system as a function of $x, y, z, \dot{x}, \dot{y}, \dot{z}$ with $\boldsymbol{OA} = x\,\boldsymbol{i}_0$, $\boldsymbol{OB} = y\,\boldsymbol{i}_0$ and $\boldsymbol{OC} = z\,\boldsymbol{i}_0$.

2) Calculate the potential energy of the system. We denote $k = 2m\,n^2$.

3) Calculate the eigenfrequencies of the system.

4) Write the Lagrange equations and stable solutions and then obtain the eigenfrequencies of the system again. Represent positions of the system for each eigenfrequency. Is it possible for the motion of the system to be periodic?

5) Write the normal coordinates α, β, γ of the system for which the kinetic energy and potential energy may be written in these new coordinates as

$$T = \frac{1}{2}\,m \sum_i \dot{Q}_i^2, \quad W = \frac{1}{2}\,m \sum_i w_i^2\,Q_i^2.$$

6) Write the equations of motion when initial conditions are

$$x - a = y - 2a = z - 3a = 0, \quad \dot{x} = u,\ \dot{y} = \dot{z} = 0.$$

This corresponds to the initial position at rest with a small impulsion $m\,u$ exerted on the first material point.

11

Solutions to Problems and Exercises

Solution to Exercise 1

For a curve of the form $y = y(x)$, the potential energy is

$$W = \int_a^b \rho\, g\, y\, \sqrt{1 + y'^2}\, dx,$$

where ρ is the wire linear mass, g is the acceleration due to gravity, and a and b are the abscissas of A and B, respectively. The length of the wire is

$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

We are back to the search for extremals of

$$a = \int_a^b G(y, y\prime)\, dx \quad \text{with} \quad G(y, y\prime) = (y - \mu)\sqrt{1 + y'^2},$$

where μ is a constant Lagrange multiplier. With the stationarity relation being

$$G - \frac{\partial G}{\partial y'}\, y' = C,$$

where C is a constant, we obtain by integration,

$$y - \mu = C \cosh\left(\frac{x - x_0}{C}\right).$$

Knowing a, b and L makes it possible to determine the constants μ, C and x_0 that represent a catenary.

Solution to Exercise 2

The curve belongs to a sphere with center A if and only if for any point M on the curve

$$\boldsymbol{AM}^2 = a^2 \quad (a \text{ constant}) \quad \Longleftrightarrow \quad \boldsymbol{AM}^\star\, \boldsymbol{T} = 0.$$

So, point A is in the plane normal to the curve. By denoting another fixed point by O, we have

$$\boldsymbol{OA} = \boldsymbol{OM} + \boldsymbol{MA} \quad \Longleftrightarrow \quad \boldsymbol{OA} = \boldsymbol{OM} + \lambda(s)\boldsymbol{N} + \mu(s)\boldsymbol{B}$$

where $\boldsymbol{N}$ and $\boldsymbol{B}$ denote the principal unit normal and the unit binormal to the curve. As the center A is fixed:

$$\frac{d\boldsymbol{OA}}{ds} = \boldsymbol{T} + \lambda'(s)\boldsymbol{N} + +\mu'(s)\boldsymbol{B} + \lambda(s)\left(-\frac{\boldsymbol{T}}{\mathrm{R}} + \frac{\boldsymbol{B}}{\tau}\right) + \mu(s)\frac{\boldsymbol{B}}{\tau} = \boldsymbol{0}.$$

Thus,

$$\lambda = \mathrm{R}, \quad \frac{d\mathrm{R}}{ds} = \frac{\mu}{\tau}, \quad \mu'(s) = -\frac{\mathrm{R}}{\tau}.$$

By eliminating μ, we obtain

$$\frac{d}{ds}\left(\tau\,\frac{d\mathrm{R}}{ds}\right) + \frac{\mathrm{R}}{\tau} = 0.$$

Let us note that from the values of λ and μ, we can find the position of the fixed point A on the plane normal to the curve at M.

Solution to Exercise 3

1) We obtain $V = \ell \displaystyle\int_a^b y(x)\,dx$.

2) We have

$$W_{total} = \sigma_1\,\ell\,[y(a) + y(b) + b - a] + \sigma_2\,\ell \int_a^b \sqrt{1 + y'^2}\,dx + \frac{1}{2}\int_a^b \rho\, g\, \ell\, y^2\,dx.$$

3) The problem is to make $a = W_{total} - \lambda V$ extremal, where λ is a constant Lagrange multiplier:

$$\frac{\delta a}{\ell} = \left[\sigma_1 + \sigma_2 \frac{y'_b}{\sqrt{1+y'^2_b}}\right] \delta y_b + \left[\sigma_1 - \sigma_2 \frac{y'_a}{\sqrt{1+y'^2_a}}\right] \delta y_a + \int_a^b \left(\rho\, g\, (y-\mu) - \sigma_2 \frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}}\right) \delta y\, dx,$$

where $\rho\, g\, \mu = \lambda$.

4) a) Let us write $y' = \mathrm{tg}\, \varphi$, where φ denotes the angle of the tangent of the meniscus with the horizontal axis Ox. Since $\sin\, \varphi = \dfrac{y'}{\sqrt{1+y'^2}}$, we obtain

$$\sin \varphi_A = -\sin \varphi_B = \frac{\sigma_1}{\sigma_2}.$$

b) Moreover,

$$\rho\, g\, (y - y_0) = \sigma_2 \frac{d \sin\, \varphi}{dx} = \sigma_2\, \cos\, \varphi \frac{d\varphi}{dx} = \sigma_2 \frac{d\varphi}{ds},$$

where s is the curvilinear abscissa of the meniscus. We deduce the Jurin law:

$$\rho\, g\, (y - y_0) = \frac{\sigma_2}{\mathrm{R}}.$$

Solution to Exercise 4

1) In the cylindrical representation, the meniscus can be written as $z = z(r)$, the bottom being the horizontal wall closing the cylindrical tube. Thus,

$$V = 2\pi \int_{r_0}^{r_1} r\, z(r)\, dr.$$

2) We obtain

$$\frac{1}{2\,\pi} W_{total} = \sigma_1\, [r_0\, z(r_0) + r_1\, z(r_1)] + \sigma_2 \int_{r_0}^{r_1} r \sqrt{1+z'^2}\, dr + \frac{1}{2} \int_{r_0}^{r_1} \rho\, g\, r\, z^2\, dr.$$

3) The problem is to make $a = W_{total} - \lambda V$ extremal, where λ is a constant Lagrange multiplier. We obtain

$$\frac{\delta a}{2\pi} = r_1 \left[\sigma_1 + \sigma_2 \frac{z_1'}{\sqrt{1+z_1'^2}}\right] \delta z_1 + r_1 \left[\sigma_1 - \sigma_2 \frac{z_0'}{\sqrt{1+z_0'^2}}\right] \delta z_0 + \int_a^b \left(\rho\, g\, r\, (z - z_m) - \sigma_2 \frac{d}{dr} \frac{r\, z'}{\sqrt{1+z'^2}}\right) \delta z\, dr,$$

where z_m replaces the Lagrange multiplier. We obtain

$$\rho\, g\, r\, (z - z_m) - \sigma_2 \frac{d}{dr} \frac{r\, z'}{\sqrt{1+z'^2}} = 0.$$

Let φ be the angle of the tangent to the meridian of the meniscus with the classical radius axis $O\boldsymbol{u}$. We have

$$\frac{d\varphi}{dr} = \frac{d\varphi}{ds}\frac{ds}{dr} = \frac{d\varphi}{ds}\frac{1}{\cos\varphi},$$

where s denotes the curvilinear abscissa of the meridian of the meniscus, and

$$\sigma_2 \left[\frac{\sin\varphi}{r} + \frac{d\varphi}{ds}\right] = \rho\, g\, (z - z_m) \quad \Longleftrightarrow \quad \sigma_2 \left[\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right] = \rho\, g\, (z - z_m).$$

This relation is valid in the case of a cylindrical tube. If the meniscus is approximated by a spherical cap with radius R,

$$\frac{2\,\sigma_2}{\mathrm{R}} = \rho\, g\, (z - z_m).$$

4) Since $\sin\varphi = \dfrac{y'}{\sqrt{1+y'^2}}$, we obtain

$$\sin\varphi_{r_0} = -\sin\varphi_{r_1} = \frac{\sigma_1}{\sigma_2}.$$

Solution to Exercise 5

1) Refer to Part 1, Chapter 2.

2) a) $\boldsymbol{N}$ and $\operatorname{grad} F$ are collinear and for the sphere $\operatorname{grad} F$ is collinear with $\boldsymbol{OM}$, hence $\boldsymbol{N} = \nu\, \boldsymbol{OM}$. Since $\boldsymbol{N}$ is collinear with $\frac{d^2\boldsymbol{OM}}{ds^2}$, we obtain $\dfrac{d^2\boldsymbol{OM}}{ds^2} \wedge \boldsymbol{OM} = \boldsymbol{0}$. There exists a vector $\boldsymbol{C}$ such that $\frac{d\boldsymbol{OM}}{ds} \wedge \boldsymbol{OM} = \boldsymbol{C}$ and M

belongs to the plane containing O and perpendicular to $\boldsymbol{C}$. Geodesics are parts of the great circles of the sphere.

b) In the spherical representation (θ, φ, R), the curvilinear abscissa ds is such that

$$ds^2 = R^2(d\varphi^2 + \cos^2\varphi\, d\theta^2),$$

hence:

$$L = \int_A^B G(\theta, \theta', \varphi)\, d\varphi \quad \text{where} \quad G(\theta, \theta', \varphi) = R\sqrt{1 + \theta'^2_\varphi\, \cos^2\, \varphi}.$$

So, the extremum condition is

$$\frac{d}{d\varphi}\left(\frac{\partial G}{\partial \theta'}\right) = 0 \quad \Longleftrightarrow \quad \frac{\theta'\, \cos^2\, \varphi}{\sqrt{1 + \theta'^2_\varphi\, \cos^2\, \varphi}} = k,$$

where k is a constant. We obtain $\lambda \operatorname{tg} \varphi = \sin(\theta - \theta_0)$, where $\lambda = \dfrac{k}{\sqrt{1 - k^2}}$ and θ_0 is a constant. By changing the spherical representation into Cartesian coordinates, we verify that this relation is the equation of a plane containing O.

3) In the cylindrical representation, the equation of a cylinder of revolution is $x = R\, \cos\theta$, $y = R\, \sin\theta, z$. The length of a curve traced on the cylinder is

$$L = \int_A^B \sqrt{R^2 + z'^2(\theta)}\, d\theta.$$

We can deduce the equation of extremals of L,

$$\frac{d}{d\theta}\left(\frac{z'(\theta)}{\sqrt{R^2 + z'^2(\theta)}}\right) = 0 \quad \Longleftrightarrow \quad \frac{z'(\theta)}{\sqrt{R^2 + z'^2(\theta)}} = k.$$

Thus, $z'(\theta) = h \implies z = h(\theta - \theta_0)$ is the equation of the cylindrical helices, where k, h and θ_0 are constants.

4) a) For $r = az$, we have

$$L = \int_\alpha^\beta G(z, z')\, d\theta, \quad \text{where} \quad G(z, z') = \sqrt{(1 + a^2)\, z'^2 + a^2\, z^2},$$

which implies the Euler equation in integrated form:

$$\frac{a^2\, z^2}{\sqrt{(1+a^2)\, z'^2 + a^2\, z^2}} = C$$

where C is a constant.

b) In the form $\theta = \theta(z)$, we obtain:

$$L = \int_a^b G(\theta', z)\, dz, \quad \text{where} \quad G(\theta', z) = \sqrt{(1+a^2) + a^2\, z^2 \theta'^2},$$

which implies the Euler equation in integrated form:

$$\frac{a^2\, z^2\, \theta'}{\sqrt{(1+a^2) + a^2\, z^2\, \theta'^2}} = K$$

where K is a constant. This differential equation has the solution

$$\frac{K}{az} = \cos\left(\frac{a(\theta - \theta_0)}{\sqrt{1+a^2}}\right).$$

Note that case (a) leads to the same representation.

5) a) The first fundamental form

$$ds^2 = E\, du^2 + 2F\, du dv + G\, dv^2$$

corresponds to $E = (\boldsymbol{M}'_u)^2, F = \boldsymbol{M}'_u.\boldsymbol{M}'_v$ and $G = (\boldsymbol{M}'_v)^2$. If the coordinate lines are orthogonal, then it corresponds to $F = 0$.

b) For $ds^2 = \dfrac{du^2 + dv^2}{u^2}$, $L = \displaystyle\int_{u_0}^{u_1} G(u, v'_u)\, du$, where $G(u, v'_u) = \dfrac{\sqrt{1 + v'^2_u}}{u}$.

By integration, the associated Euler equation is

$$v'_u = K\, u \sqrt{1 + (v'_u)^2},$$

which yields

$$v = v_0 - \frac{\sqrt{1 - K^2 u^2}}{K},$$

where K and v_0 are constants. The images in plane (Π) are circles of equations:

$$u^2 + (v - v_0)^2 = \frac{1}{K^2}.$$

Solution to Exercise 6

1) $\boldsymbol{N}$ is collinear with $\operatorname{grad} F$ and gives the direction of the normal to the surface. The normal meets and is perpendicular to the axis Oz. Thus, $\boldsymbol{N}$ is perpendicular to Oz, a property that characterizes helices. This result could be predicted: the cylinder is a surface that can be developed into a plane, and the development of helices gives straight lines in the plane.

2) For any surface of revolution written as $z = f(r), \theta = g(r)$, we have

$$ds^2 = \left[1 + f'^2(r)\right] dr^2 + r^2\, d\theta^2.$$

The Euler equation associated with $L = \displaystyle\int_{s_0}^{s_1} ds$ is integrated as

$$\frac{r^2\, \dot{\theta}}{\sqrt{1 + f'^2(r) + r^2\, g'(r)^2}} = K \quad \Longleftrightarrow \quad r^2 \frac{d\theta}{ds} = K,$$

where K is a constant. For a cylinder $r = R$, R is a constant, and we obtain $K\,(s - s_0) = \theta - \theta_0$, which represents the helices.

Solution to Exercise 7

1) a) The curve is defined by the representation $z = z(x)$. We obtain

$$S = 2\pi \int_A^B G(z, z', x) dx \quad \text{where} \quad G(z, z', x) = x\sqrt{1 + z'^2}.$$

The integrated Euler equation yields

$$\frac{dz}{dx} = \frac{a}{\sqrt{x^2 - a^2}},$$

where a is a constant. We obtain

$$z - z_0 = a \operatorname{Argch}\left(\frac{x}{a}\right),$$

where z_0 is another constant.

b) Note that it is possible to use a geometric method by writing

$$S = 2\pi \int_A^B x\, ds = 2\pi \int_A^B x\, \boldsymbol{T}^\star\, d\boldsymbol{M}.$$

2) The volume is $V = \displaystyle\int_{z_0}^{z_1} \pi\, x^2\, dz$, and we are led to find the extremals of $a = S - 2\lambda\, V$, where λ is a constant Lagrange multiplier. We obtain the integrated Euler equation:

$$x^2 - \frac{2\lambda\, x}{\sqrt{1 + x'^2}} = C \quad \text{with} \quad x = x(z),$$

where C is a constant. We obtain arcs of circles of equations:

$$(z - z_0)^2 + x^2 = 4\,\lambda^2.$$

Solution to Exercise 8

The calculations are based on an analytical or geometric review of the study of the optical path of light presented in this book, followed by the case where the optical index is $n = e^y$.

Solution to Exercise 9

A. 1) The result comes from $(ds)^2 = (dz)^2 + (dr)^2 + r^2(d\theta)^2$ written in the representation $r = f(z)$ and $\theta = \theta(z)$:

$$L = \int_{z_0}^{z_1} \sqrt{1 + f'^2(z) + f^2(z)\, \theta'^2}\, dz.$$

2) The potential energy W of the wire is

$$W = \int_A^B \rho\, g\, z\, ds = \int_{z_0}^{z_1} \rho\, g\, z\, \sqrt{1 + f'^2(z) + f^2(z)\, \theta'^2}\, dz.$$

By writing the extremals of $W - \lambda L$, where λ is a constant Lagrange multiplier, we obtain

$$(z - \lambda)\, r^2\, \frac{d\theta}{ds} = C,$$

where C is a constant.

3) We obtain

$$\frac{(z - \lambda)\, f^2(z)}{\sqrt{(1 + f'^2)\, z'^2 + f^2}} = C.$$

4) The equilibrium curve on the circular cylinder of radius R is

$$z - \lambda = \frac{K}{R} \cosh\left(\frac{R^2}{K}\,(\theta - \theta_0)\right),$$

where K is a constant. These are catenaries, or chainettes, on the cylinder.

B. 1) We have

$$L = \int_{(C)} \boldsymbol{T}^\star\, d\boldsymbol{M} \quad \text{and} \quad W = \int_{(C)} \rho\, g\, z\, \boldsymbol{T}^\star\, d\boldsymbol{M}.$$

The extremals of $a = W - \lambda L$, where λ is a constant Lagrange multiplier, lead to the variation of curvilinear integrals:

$$(z - h)\frac{d\boldsymbol{T}}{ds} - (1 - \boldsymbol{T}\,\boldsymbol{T}^\star)\,\mathrm{grad}\,(z - h) - \mu\,\mathrm{grad}\,F = 0,$$

where $F(M) = 0$ is the surface equation.

2) a) We obtain

$$\boldsymbol{T} = \frac{r'(z)\boldsymbol{u} + r\,\theta'(z)\boldsymbol{v} + \boldsymbol{k}}{\sqrt{1 + f'^2(z) + f^2(z)\,\theta'^2}}$$

and

$$(z - h)\,\frac{d\boldsymbol{T}}{ds} - (1 - \boldsymbol{T}\,\boldsymbol{T}^\star)\,\mathrm{grad}\,(z - h) - \mu\,\boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ is the unit normal to surface (S). In the basis $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{k})$, the matrix of projection is

$$\mathbf{1} - \boldsymbol{T}\boldsymbol{T}^{\star} = \begin{bmatrix} 1 - r'^2 \left(\frac{dz}{ds}\right)^2 & -r\, r' \frac{dz}{ds}\frac{d\theta}{ds} & -r' \left(\frac{dz}{ds}\right)^2 \\ -r\, r' \frac{dz}{ds}\frac{d\theta}{ds} & 1 - r^2 \left(\frac{d\theta}{ds}\right)^2 & -r \frac{dz}{ds}\frac{d\theta}{ds} \\ -r' \left(\frac{dz}{ds}\right)^2 & -r \frac{dz}{ds}\frac{d\theta}{ds} & 1 - \left(\frac{dz}{ds}\right)^2 \end{bmatrix}$$

and

$$\frac{d\boldsymbol{T}}{ds} = \begin{bmatrix} r'' \left(\frac{dz}{ds}\right)^2 + r' \frac{d^2 z}{ds^2} - r \left(\frac{d\theta}{ds}\right)^2 \\ 2\, r' \frac{d\theta}{ds}\frac{dz}{ds} + r \frac{d^2\theta}{ds^2} \\ \frac{d^2 z}{ds^2} \end{bmatrix}$$

b) i)

$$\boldsymbol{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v_2} = \frac{1}{\sqrt{1 + r'^2}} \begin{bmatrix} r' \\ 0 \\ 1 \end{bmatrix}.$$

ii) The unit normal to (S) is $\boldsymbol{n} = \dfrac{1}{\sqrt{1 + r'^2}} \begin{bmatrix} 1 \\ 0 \\ -r' \end{bmatrix}$.

3) Let us consider the simplest of the three relations already given in (1). The second component of the vectorial relation given in (1) becomes:

$$-2\,(z - h)\, r' \frac{d\theta}{ds}\frac{dz}{ds} - r\,(z - h)\frac{d^2\theta}{ds^2} - r \frac{d\theta}{ds}\frac{dz}{ds} = 0.$$

By multiplying by r and integrating, we obtain:

$$(z - h)\, r^2 \frac{d\theta}{ds} = Cte.$$

Solution to Exercise 10

Exercise 10 is a particular case of the problem given in Exercise 9. We can refer to the solution to Exercise 9 and consider the case of a cone of revolution.

Solution to Exercise 11

1) The integral $\displaystyle\int_{\widehat{AB}} \frac{ds}{x^m} = \int_{\widehat{AB}} \frac{\boldsymbol{T}^\star\, d\boldsymbol{M}}{x^m}$ is unchanged by translation parallel to the axis Oy. The translations are represented as

$$\mathcal{T}_\alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y+\alpha \end{bmatrix}.$$

They form a Lie group of infinitesimal displacement $\boldsymbol{W} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and Noether's theorem is applicable:

$$\frac{\boldsymbol{T}^\star\, \boldsymbol{W}}{x^m} = Cte \quad \Longleftrightarrow \quad \frac{1}{x^m} \sin\,\alpha = Cte \quad \Longleftrightarrow \quad x^m = a^m\, \sin\,\alpha,$$

where a is a constant.

2) From $x = a\ \sin^r \alpha$ and $dy = \operatorname{tg} \alpha\, dx$, we deduce $y = y_0 + r \displaystyle\int_{\frac{\pi}{2}}^{\alpha} x\, d\alpha$, where y_0 is the value of y for $\alpha = \dfrac{\pi}{2}$. For $m = 0$, $\sin\,\alpha = Cte$. The curves are lines of the slope α.

3) For $m = 1$, we obtain $x = a \sin\,\alpha,\ y - y_0 = -a \cos\,\alpha$, hence

$$x^2 + (y - y_0)^2 = a^2,$$

corresponding to arcs of circles.

For $m = \dfrac{1}{2}$, we obtain arcs of cycloids:

$$x = \frac{a}{2}\,(1 - \cos\,2\,\alpha)\,, \quad y - y_0 = \frac{a}{2}\,(2\,\alpha - \sin\,2\,\alpha)\,.$$

For $m = -1$, we obtain $x = a \cosh\left(\dfrac{y - y_0}{a}\right)$, which is the equation of catenary.

For $m = -\dfrac{1}{2}$, we obtain arcs of parabolas $x - a = \dfrac{1}{4a}\,(y - y_0)^2$.

4) For a given m,

$$\begin{cases} x = a \sin^{\left(\frac{1}{m}\right)} \alpha \\ \\ y = y_0 + \dfrac{a}{m} \displaystyle\int_{\frac{\pi}{2}}^{\alpha} \sin^{\left(\frac{1}{m}\right)} u \, du \end{cases}$$

The curves are deduced from one of the above by the composition of a translation relative to Oy and a homothety of center O and ratio a. The change of α into $\pi - \alpha$ leaves x unchanged and $y - y_0$ changed to its opposite; we obtain a symmetry relative to Oy.

5) The previous curves are such that $y' \, x'' - x' \, y'' < 0$ and the concavity is turned towards $x < 0$.

The remaining questions (5) and (6) can be easily solved.

Solution to Exercise 12

1) We choose $t = t(x)$ and $y = y(x)$. Therefore,

$$I = \int_{t^{-1}(0)}^{t^{-1}(T)} G(t, t', y, y') \, t' \, dx, \quad \text{where} \quad G(t, t', y, y') = \sqrt{x^2 t'^2 - 1 - y'^2}.$$

The Euler equations are:

$$\frac{d}{dx}\left(\frac{\partial G}{\partial t'}\right) = 0 \iff \frac{\partial G}{\partial t'} = Cte \quad \text{and} \quad \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0 \iff \frac{\partial G}{\partial y'} = Cte,$$

or

$$\frac{x^2 \, t'}{\sqrt{x^2 t'^2 - 1 - y'^2}} = \alpha \quad \text{and} \quad \frac{y'}{\sqrt{x^2 t'^2 - 1 - y'^2}} = \beta,$$

where α and β are constants. By eliminating y' between the two first integrals, we obtain

$$x^2 \, t'^2 \left(1 - \frac{x^2}{a^2}\right) = 1 \quad \text{with} \quad a^2 = \frac{\alpha^2}{1 + \beta^2}.$$

We finally obtain

$$x = \frac{a}{\cosh(t - t_0)}, \quad y = y_0 + b \tanh(t - t_0).$$

2) a) The previous results yield

$$x = \frac{\cosh\left(\frac{T}{2}\right)}{\cosh\left(t - \frac{T}{2}\right)}, \quad y = \frac{\tanh\left(t - \frac{T}{2}\right)}{\tanh\left(\frac{T}{2}\right)},$$

which can be written as

$$\frac{x^2}{a^2} + (a^2 - 1)\frac{y^2}{a^2} = 1, \quad \text{where} \quad a = \cosh\left(\frac{T}{2}\right),$$

corresponding to an ellipse.

b) The time displacement is minimum for $x^2 - v^2 = 0$. Since

$$x^2 - v^2 = \frac{\cosh^2\left(\frac{T}{2}\right)}{\cosh^4\left(t - \frac{T}{2}\right)} \frac{\left[\sinh^2\frac{T}{2} - 1\right]}{\sinh^2\left(\frac{T}{2}\right)},$$

we must have $T > 2\,\mathrm{Argsh}\,1$. The limit value corresponding to time lapse is $T_\ell = 2\,\mathrm{Argsh}\,1$, for which $a = \sqrt{2}$ and the trajectory is the circle $x^2 + y^2 = 2$.

3) Since $v^2 = \left(\frac{ds}{dt}\right)^2$, $I = \int_0^T \sqrt{x^2\,(dt)^2 - (ds)^2}$. For $x^2\,(dt)^2 = (ds)^2$, we obtain $dt = \frac{ds}{x}$ and $T = \int_0^T dt = \int_{s_0}^{s_1} \frac{ds}{x}$. The trajectories have been studied previously. In the representation $y = f(x)$, they are those associated with the extremal of $\int_{x_0}^{x_1} \frac{\sqrt{1+f'^2(x)}}{x}\,dx$.

Solution to Exercise 13

1) Derivatives being taken with respect to variable x, the Euler equation can be written as

$$2\,y\,y' - x\,y\,y'' - 2\,x\,y'^2 = 0$$

2) The Hamiltonian is $4\,p\,q - x^2\,y^2$ and yields the Hamilton equations

$$\begin{cases} \dfrac{dx}{d\tau} = 4\,q, & \dfrac{dp}{d\tau} = 2\,x\,y^2, \\ \dfrac{dy}{d\tau} = 4\,p, & \dfrac{dq}{d\tau} = 2\,x^2\,y. \end{cases}$$

3) a is invariant by transformation

$$\mathcal{T}_t \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t\,x \\ \dfrac{y}{t} \end{bmatrix}.$$

However, as the identity element of the group is obtained for $t = 1$, we choose a Lie group with the identity element 0 by writing $t = e^\ell$. We obtain the infinitesimal displacement $\boldsymbol{W} = \dfrac{d\boldsymbol{M}}{d\ell} = \begin{bmatrix} e^\ell\,x_0 \\ -e^\ell\,y_0 \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, and the first integral $u = [p,q]\begin{bmatrix} x \\ -y \end{bmatrix} = p\,x - q\,y$. From $p = \dfrac{1}{2}\,x\,y\dfrac{y'_\tau}{\sqrt{x'_\tau}}$, $q = \dfrac{1}{2}\,x\,y\dfrac{x'_\tau}{\sqrt{x'_\tau}}$ and from the first integral, we deduce

$$2\,u\sqrt{y'} - x^2\,y\,y' + x\,y^2 = 0.$$

4) We have $[u,H]_{(x,y,p,q)} = 0$ and as (1) is verified, (3) is therefore verified.

5) The Jacobi equation $\dfrac{\partial S}{\partial t} + H = 0$ is integrated in the form

$$S(x,y) = \frac{1}{6}\left(k\,x^3 + \frac{y^3}{k}\right)$$

where k is a constant. We obtain the extremals in the form

$$y = \sqrt[3]{\lambda^2\,x^3 + \mu}$$

where λ and μ are constants.

6) From (4), it can be immediately verified that v with value $\dfrac{\lambda}{2}$ is a constant along the motion. From the expression $\dfrac{1}{x^2}\,p + 0\,q = Cte$, we deduce $\dfrac{d\boldsymbol{M}}{dr} = \begin{bmatrix} \dfrac{1}{x^2} \\ 0 \end{bmatrix}$.

Consequently,

$$\begin{cases} \dfrac{dx}{dr} = \dfrac{1}{x^2} \\ \\ \dfrac{dy}{dr} = 0 \end{cases} \Longrightarrow \begin{cases} x^3 = x_0^3 + s, \\ \\ y = y_0 \end{cases} \quad \text{where } s = 3\,r \text{ and } \mathcal{T}_s \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{x^3 + s} \\ y \end{bmatrix}.$$

By writing $X = x^3$ and $Y = y^3$, we obtain $a = \varepsilon \displaystyle\int_{X_0}^{X_1} \sqrt{Y'_X}\, dX$, where $\varepsilon = \pm 1$, and this integral is invariant by transformation $(X \longrightarrow X + s,\ Y \longrightarrow Y)$. The Poisson bracket $[u, v]_{(x,y,p,q)} = 3\dfrac{p}{x^2}$ does not give any additional first integral; therefore, another method must be found. From $u = p\,x - q\,y$ and $v = \dfrac{p}{x^2}$, we obtain $q\,y = v\,x^3 - u$, which is of form $w\,y^3$ since $y^3 = \lambda\,x^3 + \mu$. We obtain $w = \dfrac{q}{y^2}$. By comparison with previous calculations, this result corresponds to the invariance group

$$\mathcal{T}_k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \\ \sqrt{y^3 + k} \end{bmatrix}.$$

7) The extremal curves are $b\,x^3 - y^3 = c$, where b and c are constants. In order for (x_0, y_0) and (x_1, y_1) to be connected, we must have

$$\begin{cases} b = \dfrac{y_1^3 - y_0^3}{x_0^3 - x_1^3} \\ \\ c = \dfrac{y_0^3\,x_1^3 - y_1^3\,x_0^3}{x_1^3 - x_0^3} \end{cases}$$

which requires $x_0 \neq x_1$. If $x_0 = x_1$ and $y_0 \neq y_1$, then there is no solution, otherwise $y_0 = y_1$ would correspond to the same point.

It is also essential that $y' > 0$, i.e. $b \geq 0$. The corresponding value of b is obtained by the integral $a = \displaystyle\int_{x_0}^{x_1} x\,y\,\sqrt{y'}\,dx$ for $y = \sqrt[3]{b\,x^3 + c}$. We obtain $b = \dfrac{\sqrt{(y_1^3 - y_0^3)\,(x_0^3 - x_1^3)}}{3}$.

Solution to Exercise 14

A. 1) From conservation of energy, the choice $v = 0$ for $y = 0$ yields $m\,v^2 + 2\,m\,g\,y = 0$. We obtain

$$(1 + y'^2)\left(\frac{dx}{dt}\right)^2 = -\,2\,g\,y.$$

Hence,

$$\tau = \int_0^a \sqrt{\frac{1 + y'^2}{-2gy}}\,dx = \int_0^a f(y, y')\,dx.$$

2) In order for time lapse τ to be minimum, we must have the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

which can be written as

$$\frac{d}{dy}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0 \quad \Longleftrightarrow \quad f - y'\frac{\partial f}{\partial y'} = c.$$

3) From the expression of f given in (1), we deduce

$$\frac{1}{\sqrt{1 + y'^2}} = K\sqrt{-y}$$

where K is a constant that can be integrated as

$$\begin{cases} x - x_0 = \dfrac{1}{2\,K^2}\,[2\varphi + \sin\,2\varphi] \\ \\ y = -\dfrac{1}{2\,K^2}\,[1 + \cos\,2\varphi] \end{cases}$$

which represent cycloids.

B. 1) Since $v = \dfrac{ds}{dt}$ and $\tau = \displaystyle\int_A^B dt$, we deduce

$$\tau = \int_O^A \frac{\boldsymbol{T}^\star}{v}\,d\boldsymbol{M}.$$

2) The classic relation with respect to the curvilinear integrals with $n = \dfrac{1}{v}$ leads to the relation:

$$-\frac{\rho}{v}\,\boldsymbol{N} + (1 - \boldsymbol{T}\boldsymbol{T}^{\star})\ \mathrm{grad}\ \left(\frac{1}{\mathrm{v}}\right) = 0.$$

When multiplied by $\boldsymbol{N}^{\star}$, we obtain the relation

$$\mathrm{R}\,v\,\boldsymbol{N}^{\star}\mathrm{grad}\left(\frac{1}{v}\right) = 1.$$

3) From $v = \sqrt{-2\,g\,y}$ and $\boldsymbol{N} = \dfrac{1}{\sqrt{1+y'^2}}\begin{bmatrix} -y' \\ 1 \end{bmatrix}$, we deduce the differential equation:

$$1 + y'^2 + 2\,y\,y'' = 0.$$

4) A classic integration of this differential equation by denoting $z(y) = y'$ makes it possible to obtain the cycloid of (*A.* 3) again.

Solution to Exercise 15

1) We have

$$\frac{1}{2}\,m\,\left(\frac{ds}{dt}\right)^2 + m\,g\,z = E \quad \text{where} \quad E > 0.$$

We deduce

$$\sqrt{\frac{2\,E}{m}}\,dt = \frac{ds}{\sqrt{1 - h\,z}} \quad \text{where} \quad h = \frac{m\,g}{E},$$

and since $ds = \boldsymbol{T}^{\star}\,\boldsymbol{d}M$,

$$\sqrt{\frac{2\,E}{m}}\,\tau = \int_{(C)} \boldsymbol{T}^{\star}\,\frac{\boldsymbol{dM}}{\sqrt{1 - hz}}.$$

2) We denote $n(M) = \dfrac{1}{\sqrt{1-hz}}$ and we obtain $\sqrt{\dfrac{2\,E}{m}}\,\tau = \displaystyle\int_{(C)} n\,\boldsymbol{T}^{\star}\,\boldsymbol{d}M$. Extremals of τ satisfy the geometric condition

$$\frac{dn\boldsymbol{T}}{ds} = \operatorname{grad} n.$$

Since $\operatorname{grad} n$ is collinear with $\boldsymbol{k}_0$, we obtain

$$\frac{dn\boldsymbol{T}}{ds} \wedge \boldsymbol{k}_0 = \boldsymbol{0} \iff \frac{d}{ds}\left(n\,\boldsymbol{T} \wedge \boldsymbol{k}_0\right) = \boldsymbol{0} \iff \frac{\boldsymbol{T} \wedge \boldsymbol{k}_0}{\sqrt{1-hz}} = \boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant vector.

3) For $\boldsymbol{C} = c\,\boldsymbol{j}_0$, we obtain

$$\frac{\boldsymbol{T} \wedge \boldsymbol{k}_0}{\sqrt{1-hz}} = c\,\boldsymbol{j}_0 \implies \boldsymbol{T}^{\star}\,\frac{\boldsymbol{T} \wedge \boldsymbol{k}_0}{\sqrt{1-hz}} = c\,\boldsymbol{T}^{\star}\,\boldsymbol{j}_0 \implies c\,\varphi = 0 \implies \varphi = 0.$$

Thus, $\rho = \dfrac{1}{\mathrm{R}} = 0$. The curve (C) is the line segment AH. Point M meets the plane (Π) at point H and

$$\sqrt{\frac{2\,E}{m}}\,\tau = \int_{(C)} ds = \ell \implies \tau = \sqrt{\frac{m}{2\,E}}\,\ell.$$

Solution to Exercise 16

1) We immediately obtain

$$\begin{cases} x = \lambda\,e^{2u} \\ \\ y = 3\,(u - u_0) \end{cases}$$

For $u = 0$, $x_0 = \lambda$ and $y_0 = -3\,u_0$, the solution is

$$\begin{cases} x = x_0\,e^{2u} \\ \\ y = 3u + y_0 \end{cases} \quad \text{or} \quad \boldsymbol{M} = \mathcal{T}_u\,(\boldsymbol{M}_0) \quad \text{with} \quad \boldsymbol{M} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \boldsymbol{M}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

2) The transformations $\mathcal{T}_u$ form a Lie group because

$$\mathcal{T}_{u'} \circ \mathcal{T}_u \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xe^{2u}e^{2u'} \\ y + 3u + 3u' \end{bmatrix} = \mathcal{T}_{u+u'} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$(\mathcal{T}_u)^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xe^{-2u} \\ y - 3u \end{bmatrix} = \mathcal{T}_{-u}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Furthermore, $\mathcal{T}_u$ goes to $\mathbf{1}$ when u goes to 0.

3) When u goes to 0, $\boldsymbol{W} = \dfrac{d\boldsymbol{M}}{du} = \begin{bmatrix} 2x \\ 3 \end{bmatrix}$.

4) These can be resolved in the same way as the previous questions.

Solution to Exercise 17

1) a) $\quad \delta\alpha = \dfrac{\partial\Gamma}{\partial \boldsymbol{X}}\,\delta\boldsymbol{X} + \dfrac{\partial\Gamma}{\partial \dot{\boldsymbol{X}}}\,\delta\dot{\boldsymbol{X}} = \dfrac{\partial G}{\partial \boldsymbol{Q}}\,\delta\boldsymbol{Q} + \dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\delta\dot{\boldsymbol{Q}}.$

We have

$$\boldsymbol{Q} = \mathcal{A}(\boldsymbol{X}) \quad \Longrightarrow \quad \delta\boldsymbol{Q} = \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\,\delta\boldsymbol{X} \quad \text{and} \quad \delta\dot{\boldsymbol{Q}} = \frac{d}{dt}\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right)\delta\boldsymbol{X} + \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\,\delta\dot{\boldsymbol{X}}.$$

Hence,

$$\delta\alpha = \left[\frac{\partial G}{\partial \boldsymbol{Q}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{d}{dt}\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right)\right]\delta\boldsymbol{X} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\,\delta\dot{\boldsymbol{X}}.$$

Therefore,

$$\frac{\partial\Gamma}{\partial \boldsymbol{X}} = \frac{\partial G}{\partial \boldsymbol{Q}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{d}{dt}\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right) \quad \text{and} \quad \frac{\partial\Gamma}{\partial \dot{\boldsymbol{X}}} = \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}.$$

b) Without moving the ends of (γ) and (C), calculate the variations of b:

$$\delta b = \int_{(\gamma)} \left(\frac{\partial\Gamma}{\partial \boldsymbol{X}}\,\delta\boldsymbol{X} + \frac{\partial\Gamma}{\partial \dot{\boldsymbol{X}}}\,\delta\dot{\boldsymbol{X}}\right) dt =$$

$$\int_{(\gamma)} \left[\frac{\partial G}{\partial \boldsymbol{Q}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{d}{dt}\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right)\delta\boldsymbol{X} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\,\delta\dot{\boldsymbol{X}}\right] dt.$$

By integrating by parts, without variations of extremities of (γ),

$$\delta b = \int_{(\gamma)} \left[\frac{\partial G}{\partial \boldsymbol{Q}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}} + \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{d}{dt}\left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right) - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\,\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\right)\right]\delta\boldsymbol{X}\,dt.$$

Finally,

$$\delta b = \int_{(\gamma)} \left[\frac{\partial G}{\partial \boldsymbol{Q}} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\right)\right] \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{X}}\, \delta \boldsymbol{X}\, dt = \int_{(C)} \left[\frac{\partial G}{\partial \boldsymbol{Q}} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\right)\right] \delta \boldsymbol{Q}\, dt = \delta a.$$

Thus, (C) is an extremal curve of a that implies that $(\gamma) = \mathcal{A}(C)$ is an extremal curve of b.

c) If $\mathcal{A}$ is a diffeomorphism, the mapping $\mathcal{A}^{-1}$ makes it possible to prove the reciprocal property of (b), and (γ) is an extremal curve of b if and only if $(C) = \mathcal{A}^{-1}(\gamma)$ is an extremal curve of a.

2) We have

$$\delta a = \int_{(C)} \left[\frac{\partial G}{\partial \boldsymbol{Q}} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\right)\right] \delta \boldsymbol{Q}\, dt + \left[\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}\, \delta \boldsymbol{Q}\right]_{\boldsymbol{A}}^{\boldsymbol{B}},$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are extremities of $(\mathcal{C})$. However, this result is independent of the end points chosen on $(\mathcal{C})$, and if we replace $(\mathcal{C})$ with $(\mathcal{C}')$, a curve contained in $(\mathcal{C})$ with the ends $\boldsymbol{A}'$ and $\boldsymbol{B}'$, we obtain the same result. Let us choose:

a) $\delta \boldsymbol{Q} = \boldsymbol{W}(\boldsymbol{Q})\, du$, where $\boldsymbol{W}(\boldsymbol{Q})$ is the infinitesimal displacement of group $\{\mathcal{T}_u,\ u \in \mathbb{R}\}$;

b) a curve $(\mathcal{C})$, extremal of a.

On the one hand, $\delta \boldsymbol{Q} = \boldsymbol{W}(\boldsymbol{Q})\, du$ implies $\delta a = 0$, since a is invariant by the group $\{\mathcal{T}_u,\ u \in \mathbb{R}\}$. The curve $(\mathcal{C})$ is an extremal of a; therefore, $\dfrac{\partial G}{\partial \boldsymbol{Q}} - \dfrac{d}{dt}\left(\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}\right) = 0$.

On the other hand, it can be deduced from $\delta \boldsymbol{B} = \boldsymbol{W}(\boldsymbol{B})\, du$ and $\delta \boldsymbol{A} = \boldsymbol{W}(\boldsymbol{A})\, du$ that

$$\frac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{B})\, \delta \boldsymbol{B}\, du - \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{A})\, \delta \boldsymbol{A}\, du = 0 \ \text{ then } \ \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{B})\, \delta \boldsymbol{B} - \frac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{A})\, \delta \boldsymbol{A} = 0,$$

where $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{A})$ and $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}(\boldsymbol{B})$ are the values of $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}$ at $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

Points $\boldsymbol{A}$ and $\boldsymbol{B}$ play no particular role in the proof, and $\dfrac{\partial G}{\partial \dot{\boldsymbol{Q}}}\, \boldsymbol{W}$ is independent of the point chosen on extremal curve $(\mathcal{C})$.

Solution to Exercise 18

1) The Lagrangian is

$$G = \frac{1}{2}\, m_2\, \dot{r}^2 + \frac{1}{2}\, m_1 \left(\dot{r}^2 + r^2\, \dot{\theta}^2 \right) - m_2\, g\, r.$$

From the Lagrange equations, we obtain

$$r^2\, \dot{\theta} = C \quad \text{and} \quad \dot{r}^2 = A - 2B\, r - \frac{C}{r^2}$$

where A, B and C are constants depending on initial conditions and on m_1, m_2, g.

2) If $A < 2\sqrt{CB} + B$, no motion is possible. If $A = 2\sqrt{CB} + B$, the equilibrium position r_0 is stable. We deduce that $\dot{\theta} = Cte$, and we obtain a circular and uniform motion for M_1. If $A > 2\sqrt{CB} + B$, we obtain a periodic oscillation between positions r_1 and r_2, and θ varies in the same orientation as r.

Solution to Exercise 19

1) From the conservation of energy, we obtain $2T = 2E$, where E is a constant

$$\dot{x}^2 + \dot{y}^2 = 2E\, y^2.$$

The Lagrange equation with respect to x is

$$\dot{x} = p_0\, y^2,$$

where p_0 is a constant. Since the Lagrangian is invariant by translation along the x-axis, we obtain this first integral again. Since the Lagrangian is invariant in time, we have conservation of energy. Taking into account the previous first integral, we deduce

$$p_0^2\, y^4 + \dot{y}^2 = 2E\, y^2.$$

This equation becomes

$$\frac{du}{u\sqrt{1-u^2}} = \varepsilon\sqrt{2E}dt \quad \text{where} \quad u = \frac{p_0}{\sqrt{2E}}y, \quad \text{with} \quad \varepsilon = \pm 1.$$

By writing $u = \sin\varphi$, we obtain

$$\frac{d\varphi}{\sin\varphi} = \varepsilon\sqrt{2E}dt,$$

which can be integrated in the form

$$y = \frac{\sqrt{2E}}{p_0} \sin \varphi.$$

Finally,

$$y = 2\frac{\sqrt{2E}}{p_0} \frac{k e^{\varepsilon\sqrt{2E}\, t}}{1 + k^2 e^{2\varepsilon\sqrt{2E}\, t}},$$

where k is a constant, and

$$x - x_0 = -2\varepsilon \frac{\sqrt{2E}}{p_0} \frac{1}{1 + k^2 e^{2\varepsilon\sqrt{2E}t}},$$

where x_0 is another constant.

2) The conjugate variables are $p = \dfrac{\dot{x}}{y^2}$, $q = \dfrac{\dot{y}}{y^2}$ and they yield the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + q^2\right) y^2.$$

The Hamilton equations can be written in the form

$$\frac{dx}{p\, y^2} = \frac{dy}{q\, y^2} = -\frac{dp}{0} = -\frac{dq}{(p^2 + q^2)\, y} = \frac{dh}{0} = dt,$$

with the convention that the numerator is zero when the denominator is zero. We deduce $p = p_0$, $h = -E$ and $(p_0^2 + q^2)y^2 = 2E$, and as in (1), we obtain the variables x and y as functions of t. From $p\, dx + q\, dy + y\, dq = 0$ and since $p = p_0$, we obtain

$$p_0\,(x - \beta) + q\, y = 0.$$

Since $q^2\, y^2 = 2E - p_0\, y^2$, we obtain

$$(x - \beta)^2 + y^2 = \frac{2E}{p_0^2}.$$

The trajectories are circles centered on the axis Ox.

3) Since $H = \dfrac{1}{2}\left(p^2 + q^2\right) y^2$, x and t are secondary variables. We research a generating function in the form $S = -E\, t + p_0\, x + S_1(y)$, which satisfies the partial

differential equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] y^2 = 0.$$

We obtain

$$\left(\frac{dS_1}{dy}\right)^2 = \frac{2E}{y^2} - p_0^2,$$

and hence, $S = -E\,t + p_0\,x + \varepsilon \int \sqrt{\frac{2E}{y^2} - p_0^2}\,dy$. The equations of motion are

$$\frac{\partial S}{\partial E} = -t_0 \quad \text{and} \quad \frac{\partial S}{\partial p_0} = \beta.$$

We obtain

$$\frac{\partial S}{\partial E} = -t + \varepsilon \int \frac{dy}{y^2\sqrt{\frac{2E}{y^2} - p_0^2}} = -t_0,$$

and consequently,

$$\varepsilon\sqrt{2E}(t - t_0) = \int \frac{dy}{y^2\sqrt{\frac{2E}{y^2} - p_0^2}} = -t_0,$$

which gives y as a function of time. Furthermore,

$$\frac{\partial S}{\partial p_0} = x - \varepsilon \int \frac{p_0\,dy}{\sqrt{\frac{2E}{y^2} - p_0^2}} = \beta,$$

and consequently,

$$x - \beta = -\frac{\varepsilon}{p_0}\sqrt{2E - p_0^2\,y^2},$$

which is the expression obtained in (2).

4) Transformation $\mathcal{T}(x,y) = (\alpha\, x, \alpha\, y)$ keeps the Lagrangian invariant. We write $\alpha = e^u$, and the infinitesimal displacement is $\boldsymbol{W} = \begin{bmatrix} x \\ y \end{bmatrix}$. According to Noether's theorem, $p\, x + q\, y = a$, where a is a constant. However, $p = p_0$ and $\left(p^2 + q^2\right)\, y^2 = 2E$. So, we obtain

$$p_0\, y^2 = 2E - (a - p_0\, x)^2 = 2E - p_0^2 (x - \beta)^2 \quad \text{where} \quad a = p_0\, \beta.$$

Finally,

$$(x - \beta)^2 + y^2 = \frac{2E}{p_0^2},$$

which are the circles found in (2). The trajectories clearly depend on three constants associated with the initial conditions.

Solution to Exercise 20

1) Since the equation of the meridian is $z = f(r)$, the associated area can be written as

$$S = \int_{r_0}^{r_1} 2\,\pi\, r\, dr = 2\pi \int_{r_0}^{r_1} G(z, z', r)\, dr \quad \text{with} \quad G(z, z', r) = r\, \sqrt{1 + z'^2},$$

where r_0 and r_1 are the radii of the two circles. The Euler equation is

$$\frac{d}{dr}\left(\frac{\partial G}{\partial z'}\right) - \frac{\partial G}{\partial z} = 0, \quad \text{which is integrated as} \quad \frac{\partial G}{\partial z'} = k,$$

where k is a constant. Hence, $z'^2 = \dfrac{k^2}{r^2 - k^2}$. Integrals are catenaries of the form

$$r = b \cosh\left(\frac{z - z_0}{b}\right)$$

where z_0 and b are two constants determined by the circles.

2) The conjugate variable of z is $p = \dfrac{\partial G}{\partial z'} = -\dfrac{r\, z'}{\sqrt{1 + z'^2}}$ and gives the Hamiltonian

$$H = p\, z' - G = \frac{r}{\sqrt{1 + z'^2}} \quad \Longrightarrow \quad H = -\sqrt{r^2 - p^2}.$$

We deduce the Hamilton equations

$$z' = \frac{\partial H}{\partial p} = -\frac{p}{\sqrt{r^2 - p^2}} \quad \text{and} \quad p' = -\frac{\partial H}{\partial z} = 0 \implies p = k$$

and obtain the equation in (1) again.

Let us find $S = S_1(r) + S_2(z)$. The Jacobi equation is

$$H(z, \frac{\partial S}{\partial z}, r) + \frac{\partial S}{\partial r} = 0 \quad \Longrightarrow \quad r^2 - \left(\frac{dS_1}{dr}\right)^2 - \left(\frac{dS_2}{dz}\right)^2 = 0.$$

We deduce

$$\frac{dS_1}{dr} - r^2 + \beta^2 = 0 \quad \text{and} \quad \frac{dS_2}{dz} = \beta,$$

where β is a constant. Finally, $S = \beta\, z + \displaystyle\int_{r_0}^{r} \sqrt{r^2 - \beta^2}\, dr$; therefore, the "motion equations" (meridian equations) are

$$\frac{\partial S}{\partial \beta} + \gamma = 0 \quad \Longrightarrow \quad z = \int_{r_0}^{r} \frac{\beta\, dr}{r^2 - \beta^2} + \gamma,$$

where γ is a constant, which corresponds to the result in (1) since $\dfrac{dz}{dr} = \dfrac{\beta}{\sqrt{r^2 - \beta^2}}$.

3) According to (1), the meridians are obtained as extremals of $a = \displaystyle\int_{(C)} n\, ds$ when $n = r$.

4) In Euclidean plane $(O, \boldsymbol{i}, \boldsymbol{k})$, $a = \displaystyle\int_{M_0}^{M_1} r\, \boldsymbol{T}^\star d\boldsymbol{M}$. This integral is invariant by translation along the axis Ox. According to Noether's theorem

$$r\, \boldsymbol{T}^\star \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e \quad \Longrightarrow \quad r\, \boldsymbol{T}^\star \boldsymbol{k} = e,$$

where e is a constant and $\boldsymbol{T} = \dfrac{1}{\sqrt{1 + z'^2}}\, \boldsymbol{i} + \dfrac{z'}{\sqrt{1 + z'^2}}\, \boldsymbol{k}$.

So, the catenaries satisfy Fermat's equation which is associated with the variation of curvilinear integrals: $\dfrac{dn\boldsymbol{T}}{ds} = \operatorname{grad} n$.

Because $\operatorname{grad} n = \operatorname{grad} r = \boldsymbol{i}$, we obtain:

$$n\,\frac{d\boldsymbol{T}}{ds} + \frac{dn}{ds}\,\boldsymbol{T} = \boldsymbol{i}.$$

The angle between the tangent to the meridian and $\boldsymbol{i}$ (unit vector of the axis r) being denoted by φ, the relations $dr = ds\,\cos\varphi,\ dz = ds\,\sin\varphi$ imply

$$r\,\frac{\boldsymbol{N}}{\mathrm{R}} + \cos\varphi\,\boldsymbol{T} = \boldsymbol{i},$$

where $\boldsymbol{N}$ is the unit normal to the meridian and R is the radius of curvature. By projection of the relation on $\boldsymbol{N}$, we obtain $\dfrac{r}{\mathrm{R}} = \sin\varphi = \cos\left(\dfrac{\pi}{2} - \varphi\right)$.

Using a diagram, we can see that at any point M, the curvature center of the meridian is symmetric to the intersection point between the normal and the revolution axis $O\boldsymbol{k}$.

Solution to Exercise 21

1) $\quad L = \dfrac{1}{2}\left(y^2\,\dot{x}^2 + x^2\,\dot{y}^2\right), \quad H(x,y,p,q) = \dfrac{p^2}{y^2} + \dfrac{q^2}{x^2}.$

Lagrange equations:

$$\begin{cases} 2\,y\,\dot{x}\,\dot{y} + \ddot{x}\,y^2 - x\,\dot{y}^2 = 0, \\ 2\,x\,\dot{x}\,\dot{y} + \ddot{y}\,x^2 - y\,\dot{x}^2 = 0. \end{cases}$$

Hamilton equations:

$$\begin{cases} \dot{x} = \dfrac{2p}{y^2}, \qquad \dot{p} = \dfrac{q^2}{x^3}, \\[2ex] \dot{y} = \dfrac{2q}{x^2}, \qquad \dot{q} = \dfrac{p^2}{y^3}. \end{cases}$$

Jacobi equation:

$$S = -E\,t + S_1(x,y) \quad \text{and} \quad \frac{1}{y^2}\left(\frac{\partial S_1}{\partial x}\right)^2 + \frac{1}{x^2}\left(\frac{\partial S_1}{\partial y}\right)^2 = 2E.$$

2) $u = x\,p - y\,q$. By using the Hamilton equations, it can be verified that $\dot{u} = 0$.

3) By using the Hamilton equations, it can be verified that $\dot{v} = 0$. By using Poisson brackets, we obtain:

$$w = \frac{p}{y}\,\sin\left(\mathrm{Log}\,\frac{x}{y}\right) - \frac{q}{x}\,\cos\left(\mathrm{Log}\,\frac{x}{y}\right).$$

4) Equation of the trajectories:

$$xy\left[A\,\cos\left(\mathrm{Log}\,\frac{x}{y}\right) + B\,\sin\left(\mathrm{Log}\,\frac{x}{y}\right)\right] = u,$$

where u, A, B are constants.

5) New Jacobi equation:

$$\xi^2 f'(\xi)^2 + \eta^2 g'(\eta)^2 = E\xi^2.$$

We deduce

$$S = -E\,t + \frac{c}{2}\,\mathrm{Log}\eta + \int \sqrt{\frac{E}{2} - \frac{c^2}{4\xi^2}}\,d\xi$$

and we can obtain equations of time law and trajectories.

Solution to Exercise 22

1) Hamiltonian:

$$H(x, y, p, q) = \frac{1}{2}\left[\frac{x}{y}\,p^2 + \frac{y}{x}\,q^2\right].$$

Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{y}{x}\,\dot{x}\right) - \dfrac{1}{2}\left[\dfrac{\dot{y}^2}{y} - \dfrac{y\,\dot{x}^2}{x^2}\right] = 0, \\[2ex] \dfrac{d}{dt}\left(\dfrac{x}{y}\,\dot{y}\right) - \dfrac{1}{2}\left[\dfrac{\dot{x}^2}{x} - \dfrac{x\,\dot{y}^2}{y^2}\right] = 0. \end{cases}$$

Hamilton equations:

$$\begin{cases} \dot{x} = \dfrac{x}{y}\,p, \qquad \dot{p} = -\dfrac{1}{2}\left[\dfrac{p^2}{y} - \dfrac{y}{x^2}\,q^2\right], \\ \dot{y} = \dfrac{y}{x}\,q, \qquad \dot{q} = -\dfrac{1}{2}\left[\dfrac{q^2}{x} - \dfrac{x}{y^2}\,p^2\right]. \end{cases}$$

Jacobi equation: noting that t is a secondary parameter,

$$S = -E\,t + S_1(x,y) \quad \text{and} \quad \frac{x}{y}\left(\frac{\partial S_1}{\partial x}\right)^2 + \frac{y}{x}\left(\frac{\partial S_1}{\partial y}\right)^2 = 2E.$$

2) H is independent of t:

$$\frac{1}{2}\left[\frac{x}{y}\,p^2 + \frac{y}{x}\,q^2\right] = E.$$

3) $\mathcal{T}_\alpha$ is a Lie group with the infinitesimal displacement $\begin{bmatrix} x \\ -y \end{bmatrix}$; we deduce the first integral $u = p\,x - q\,y$. The Hamilton equations immediately imply $\dot{u} = 0$.

4) The value is $n = 2$. $\mathcal{T}_\lambda$ is a Lie group of infinitesimal displacement $\begin{bmatrix} x \\ y \\ 2t \end{bmatrix}$; we deduce the first integral $v = p\,x + q\,y + 2h\,t$. The Hamilton equations imply

$$\dot{v} = 0, \quad p\,x + q\,y = 2c, \quad p\,x + q\,y = 2E(t - t_0).$$

5) The relation is

$$E\,xy = c^2 + E^2(t - t_0)^2.$$

6) We obtain

$$\frac{d\theta}{dt} = \frac{cE}{c^2 + E^2(t - t_0)^2}, \quad \operatorname{tg}\theta = \frac{E}{c}\,(t - t_0) + a$$

where a is a constant.

7) The equation of the trajectories is

$$E\,x\,y\cos^2\left[\frac{1}{2}\operatorname{Log}\left(\frac{x}{y}\right)\right] = c^2.$$

8) We find

$$f'(\xi) = \frac{\sqrt{E\,\xi - c^2}}{\xi}, \quad g(\eta) = c\,\mathrm{Log}\,\eta$$

with

$$S = -E\,t + \int \frac{\sqrt{E\,\xi - c^2}}{\xi}\,d\xi + c\,\mathrm{Log}\,\eta.$$

With $\dfrac{\partial S}{\partial E} = -t_0$ and $\dfrac{\partial S}{\partial c}$ being constant, we obtain the time law and trajectories of the motion.

Solution to Exercise 23

1) Hamiltonian:

$$H(\rho, \theta, p, q) = \frac{1}{2}\left(\frac{p^2}{\rho^2} + \frac{q^2}{\rho^4}\right).$$

Lagrange equations:

$$\rho^4\,\dot\theta = C, \quad \rho^2\,\ddot\rho + \rho\,\dot\rho^2 - 2\rho^3\,\dot\theta^2 = 0.$$

Hamilton equations:

$$\begin{cases} \dot\rho = \dfrac{p}{\rho^2}, \qquad \dot\theta = \dfrac{q}{\rho^4}, \\[2ex] \dot p = \dfrac{p^2}{\rho^3} + 2\,\dfrac{q^2}{\rho^5}, \qquad \dot q = 0. \end{cases}$$

Jacobi equation:

$$\rho^4\,\frac{\partial S}{\partial t} + \frac{1}{2}\,\rho^2\left[\left(\frac{\partial S}{\partial \rho}\right)^2 + \left(\frac{\partial S}{\partial \theta}\right)^2\right] = 0, \quad \text{with} \quad S = -E\,t + C\,\theta + S_1(\rho).$$

$$S = -E\,t + C\,\theta + \varepsilon\int_{\rho_0}^{\rho} \frac{\sqrt{2E\,\rho^4 - C^2}}{\rho}\,d\rho.$$

Trajectories (equilateral hyperbolas):

$$\rho^2 \cos\left[2\left(\theta - \theta_0\right)\right] = \frac{C}{\sqrt{2}E}.$$

Time law:

$$t - t_0 = \frac{C}{4E} \operatorname{tg} 2\left(\theta - \theta_0\right).$$

2) The Lagrangian is invariant by the change of θ into $\theta + \alpha$; by using Noether's theorem, we obtain the first integral $q = C$.

3) We have $\dot{u} = 0$.

4) By using the Poisson brackets, we obtain the first integral

$$w = \frac{p}{\rho} \sin 2\theta + \frac{q}{\rho^2} \cos 2\theta.$$

5) We find $\rho^2 \cos 2\left(\theta - \theta_0\right) = \dfrac{v}{u} = Cte.$

6) Noting that t is a secondary parameter, we are back to extremals of $a = \displaystyle\int \sqrt{2E}\,\rho\, ds$, where s is the curvilinear abscissa of the trajectory, and by using Maupertuis' principle, we obtain $\rho^2 \cos\left[2\left(\theta - \theta_0\right)\right] = \dfrac{C}{\sqrt{2}E}$ again.

Solution to Exercise 24

1) $\quad 2\,H = \dfrac{x\,p^2 + y\,q^2}{x+y} + 2\,W(x,y).$

2) $\quad W(x,y) = \dfrac{W_1(x) + W_2(y)}{x+y}.$

3) $\quad H = \dfrac{1}{2}\,\dfrac{x\,p^2 + y\,q^2}{x+y}$

and the Hamilton equations are

$$\begin{cases} \dot{x} = \dfrac{x\,p}{x+y}, & \dot{p} = -\dfrac{1}{2}\,\dfrac{y(p^2-q^2)}{(x+y)^2}, \\[2ex] \dot{y} = \dfrac{y\,q}{x+y}, & \dot{q} = -\dfrac{1}{2}\,\dfrac{x(q^2-p^2)}{(x+y)^2}. \end{cases}$$

Additionally, $H = E$ (constant total energy).

4) We write $S = -E\,t + f(x) + g(y)$, and we obtain

$$x\,f'^2(x) + y\,g'^2(y) - 2E(x+y) = 0,$$

which implies

$$f(x) = \int \sqrt{2E + \frac{\alpha}{x}}\,dx \quad \text{and} \quad g(y) = \int \sqrt{2E - \frac{\alpha}{y}}\,dy,$$

where $E > 0$ and α is a constant. Hence, the time law is

$$t - t_0 = \int \frac{dx}{\sqrt{2E + \dfrac{\alpha}{x}}} + \int \frac{dy}{\sqrt{2E - \dfrac{\alpha}{y}}},$$

and the trajectories verify

$$\int \frac{dx}{x\sqrt{2E + \dfrac{\alpha}{x}}} - \int \frac{dy}{y\sqrt{2E - \dfrac{\alpha}{y}}} = \beta,$$

where β is a constant and equivalently

$$\text{Argch}\left(\frac{4E\,x}{\alpha} + 1\right) - \text{Argch}\left(\frac{4E\,y}{\alpha} - 1\right) = Cte.$$

5) Hamilton equations show that $\dot{\alpha} = 0$; therefore, α is the first integral.

6) Calculate $b(x,y)$ such that $\dot{w} = 0$ or such that Poisson bracket $[w, H]$ with respect to (x, y, p, q) is zero. We obtain $w = (p-q)\,b(x,y)$ and taking into account that $b(1,1) = 1$, we obtain $b(x,y) = \sqrt{x\,y}$. Similarly, $v = (p+q)\,\dfrac{\sqrt{x\,y}}{x+y}$.

7) Calculate $u = [v, w]$. We obtain $u = \dfrac{p\,x - q\,y}{x+y}$. We find $u^2 + v^2 = 2H$ and $[u, v] = 0$.

8) By elimination of p and q between the three first integrals u, v, w, we obtain

$$u\,(x+y) = \frac{1}{2\sqrt{x\,y}}\left[v\,(x^2 - y^2) + w\,(x+y)\right].$$

9) Hamiltonian is invariant for $n = 2$. After writing $\lambda = e^u$, we obtain a Lie group of which the infinitesimal displacement is $\boldsymbol{W} = \begin{bmatrix} x \\ y \\ 2t \end{bmatrix}$. Consequently, $p\,x + q\,y = 2E\,t$. Hence,

$$t = \frac{1}{2E}\,(p\,x + q\,y) = \frac{1}{2E}\left(x\sqrt{2E + \frac{\alpha}{x}} + y\sqrt{2E - \frac{\alpha}{y}}\right),$$

which corresponds to an integral of (I) by taking into account (II).

10) The infinitesimal displacement of group G^1 is $\boldsymbol{W}_1 = \begin{bmatrix} \dfrac{x}{x+y} \\ \dfrac{-y}{x+y} \\ 0 \end{bmatrix}$. As X is the parameter of the group, we have

$$\begin{cases} \dfrac{dx}{x} = \dfrac{dX}{x+y}, \\ \dfrac{dy}{-y} = \dfrac{dX}{x+y}, \end{cases}$$

which is integrated as

$$\begin{cases} x - y = x_0 - y_0 + X, \\ x\,y = x_0\,y_0. \end{cases}$$

11) The infinitesimal displacement of group G^2 is $\boldsymbol{W}_2 = \begin{bmatrix} \dfrac{\sqrt{x\,y}}{x+y} \\ \dfrac{\sqrt{x\,y}}{x+y} \\ 0 \end{bmatrix}$. As Y is the parameter of the group, we have

$$\frac{dx}{\sqrt{x\,y}} = \frac{dY}{x+y} = \frac{dy}{\sqrt{x\,y}}$$

which is integrated as

$$\begin{cases} x - y = x_0 - y_0, \\ 2\sqrt{x\,y} = 2\sqrt{x_0\,y_0} + Y \end{cases}$$

12) The equations obtained in 10) and 11) make it possible to obtain the change of variables $(X, Y) \to (x, y)$ for $x_0 = 0$ and $y_0 = 0$:

$$\begin{cases} x - y = X, \\ 2\sqrt{x\,y} = Y, \end{cases}$$

and we obtain the new Hamiltonian associated with X, Y.

Solution to Exercise 25

1) The equation of the meridian $r = f(z)$ reduces the number of parameters to two. In cylindrical coordinates, the parameters are z and θ. The kinetic energy and potential energy are

$$T = \frac{1}{2}\, m\, \left[\left(1 + f'^2(z)\right)\, \dot{z}^2 + f^2(z)\, \dot{\theta}^2\right] \quad \text{and} \quad W = m\, g\, z.$$

2) a) The kinetic energy is second-degree homogeneous with respect to the derivatives of parameters, and $T - W$ is independent of t. We deduce the first integral of energy $T + W = Cte$:

$$\left(1 + f'^2(z)\right)\, \dot{z}^2 + f^2(z)\, \dot{\theta}^2 + 2\, g\, z = -2\, h,$$

and the Lagrange equation for θ is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = 0 \quad \Longleftrightarrow \quad f^2(z)\, \dot{\theta} = C$$

where C is a constant associated with the law of areas.

b) The conjugate variables are

$$p = \frac{\partial G}{\partial \dot{z}} = m\, \left[1 + f'^2(z)\right]\, \dot{z} \quad \text{and} \quad q = \frac{\partial G}{\partial \dot{\theta}} = m\, f^2(z)\, \dot{\theta}$$

and the Hamiltonian is

$$H(p, q, z, \theta, t) = \frac{1}{2\, m}\left[\frac{p^2}{1 + f'^2(z)} + \frac{q^2}{f^2(z)} + 2\, m^2\, g\, z\right].$$

We deduce the Hamilton equations:

$$\frac{dp}{dt} = -m\,g + \frac{1}{m}\,\frac{p^2\,f'(z)\,f''(z)}{1+f'^2(z)} + \frac{1}{m}\,\frac{q^2\,f'(z)}{f^3(z)} \quad \text{and}$$

$$\frac{dq}{dt} = 0 \quad \Longleftrightarrow \quad m\,f^2(z)\,\dot{\theta} = C.$$

c) Hamiltonian action:

$$\int_{t_0}^{t_1} G(z,\theta,\dot{z},\dot{\theta},t)\,dt = \int_{X_0}^{X_1} \boldsymbol{Y}^\star\,d\boldsymbol{X} \quad \text{where} \quad \boldsymbol{Y}^\star = [p,q,h], \quad \text{and} \quad \boldsymbol{X} = \begin{bmatrix} z \\ \theta \\ t \end{bmatrix}$$

is invariant by translation group $\mathcal{T}_u$ such that $\mathcal{T}_u \begin{bmatrix} z \\ \theta \\ t \end{bmatrix} = \begin{bmatrix} z \\ \theta+u \\ t \end{bmatrix}$, where the infinitesimal displacement vector is $\boldsymbol{W}_\theta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. By Noether's theorem, we obtain the first integral $\boldsymbol{Y}^\star\,\boldsymbol{W}_\theta \equiv q = C$. Similarly, Hamiltonian action is invariant by translation group $\mathcal{T}_v$ such that $\mathcal{T}_v \begin{bmatrix} z \\ \theta \\ t \end{bmatrix} = \begin{bmatrix} z \\ \theta \\ t+v \end{bmatrix}$, where infinitesimal displacement is $\boldsymbol{W}_t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We obtain the first integral $\boldsymbol{Y}^\star\,\boldsymbol{W}_t \equiv h = -E$, where E is a constant corresponding to the $T+W$ value.

3) The parameter θ is secondary and associated with $\dot{\theta} = \dfrac{C}{f^2(z)}$; we define the new generating function $G_1 = G - m\,\dfrac{C}{f^2(z)}$, which, after the reinjection of the first integral $q = m\,C$, becomes

$$G_2(z,\dot{z},t) = \frac{1}{2}\left[\left(1+f'^2(z)\right)\,\dot{z}^2 + \frac{C^2}{f^2(z)} - 2\,g\,z - 2\,\frac{C^2}{f^2(z)}\right].$$

and we obtain the integrated Lagrange equation:

$$\frac{m}{2}\left[\left(1+f'^2(z)\right)\,\dot{z}^2 + \frac{C^2}{f^2(z)} - 2\,g\,z - 2\right] = Cte.$$

Similarly, t is a secondary parameter and we obtain the new generating function $G_3 = G - (-E)\,t$. Thanks to the Maupertuis principle, we return to the study of the

extremals of $\int_{t_0}^{t_1} \sqrt{2\,(E-W)}\,ds$ corresponding to the optical path of $a = \int_{s_0}^{s_1} n\,ds$, where n=$\sqrt{2\,(E - m\,g\,z)}$.

Solution to Exercise 26

1) The Hamiltonian is deduced from

$$T + W = \frac{1}{2}\,(\dot{r}^2 + r^2\,\dot{\theta}^2) - \frac{k^2}{2\,r^4}$$

with $p_r = \dot{r}$ and $p_\theta = r^2\,\dot{\theta}$, and the Hamilton equations can be written.

2) The two first integrals are the law of areas

$$r^2\,\dot{\theta} = C$$

and the conservation of energy

$$\frac{1}{2}\,(\dot{r}^2 + r^2\,\dot{\theta}^2) - \frac{k^2}{2\,r^4} = E.$$

We obtain

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{C}{r^2}\,\frac{dr}{\sqrt{\dfrac{k^2}{r^4} - \dfrac{C^2}{r^2} + 2E}} \quad \text{and} \quad t - t_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{\dfrac{k^2}{r^4} - \dfrac{C^2}{r^2} + 2E}},$$

corresponding to the equation of the trajectory and the time law, respectively.

3) For $E = 0$, we immediately obtain expressions for $F(r)$ and $G(r)$.

a) The trajectory is

$$r = \frac{k}{C}\,\cos\,(\theta - \theta_0),$$

corresponding to circles passing through O.

b) The time law associated with point O is

$$t = \frac{k^2}{C^3}\,\mathrm{Arcsin}\left(\frac{Cr}{k}\right) - \frac{kr}{2C^2}\sqrt{1 - \frac{C^2 r^2}{k^2}}.$$

When θ varies from π, time varies from $T = \dfrac{\pi\, k^2}{C^3}$ along the trajectory.

4) We obtain $\displaystyle\int_{s_0}^{s_1} n\, ds$ with $n = \sqrt{2\left(E + \frac{k^2}{r^4}\right)}$. For $E = 0$, we are back to the extremals of $\displaystyle\int_{s_0}^{s_1} \frac{\sqrt{2}\, k}{r^2}\, ds$. Writing the trajectories in the form $r = r(\theta)$, we obtain a Euler equation independent of θ, which can be integrated as

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{a^2 - r^2}},$$

where a is a constant. We obtain circles again in the planar polar representation.

5) The Lagrange equations of motion are

$$r^2\, \dot\theta = C \quad \text{and} \quad \ddot r - \frac{C^2}{r^3} + \frac{2k^2}{r^5} = 0.$$

We obtain constant r for $r_0 = \sqrt{2}\dfrac{k}{C}$ and for initial condition $\dot\theta = \dfrac{C^3}{2k^2}$ corresponding to a uniform circular motion. We deduce $E = \dfrac{C^4}{8k^2}$.

Solution to Exercise 27

1) a) We verify that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{T}_{\alpha+\beta} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \mathcal{T}_\alpha \circ \mathcal{T}_\beta \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \mathcal{T}_0 = \mathbf{1}, \quad \mathcal{T}_\alpha \circ \mathcal{T}_{-\alpha} = \mathbf{1}.$$

So, transformations $(\mathcal{T}_\alpha)_\alpha$ form an abelian Lie group.

b) For $\alpha = 0$, $x_0 = x$ and $y_0 = y$, we obtain

$$\frac{\partial x}{\partial \alpha}_{|\alpha=0} = 1 + x^2 \quad \text{and} \quad \frac{\partial y}{\partial \alpha}_{|\alpha=0} = -x\, y.$$

Consequently, the infinitesimal displacement is $\boldsymbol{W}_1(x, y) = \begin{bmatrix} 1 + x^2 \\ \\ -xy \end{bmatrix}$.

c) From Noether's theorem, the associated first integral is

$$u = (1 + x^2)p - x\, y\, q.$$

2) a) We have the relations

$$y\,dx = \frac{dy}{0} = d\alpha$$

(a zero denominator corresponds to a zero numerator), where the Lie group is associated with $\mathcal{T}_\beta$ such that

$$\mathcal{T}_\beta \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} x = x_0 + \dfrac{\beta}{y_0} \\ y = y_0 \end{cases}.$$

b) The second first integral is $v = \dfrac{p}{y}$.

3) The Poisson bracket for u and v is

$$w = [u, v] = \frac{\partial u}{\partial x}\frac{\partial v}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial y} \quad \Longrightarrow \quad w = p\,\frac{x}{y} - q.$$

This is the third first integral, the infinitesimal displacement of which associated with the third group of invariance is $\boldsymbol{W}_3(x, y) = \begin{bmatrix} \dfrac{x}{y} \\ -1 \end{bmatrix}$. We have the relations

$$\frac{y\,dx}{x} = -dy = d\alpha.$$

Hence, $y = y_0 + \alpha$ and $\dfrac{x}{x_0} = \dfrac{y_0}{y_0 - \alpha}$. The Lie group associated with $\mathcal{T}_\alpha$ is

$$\mathcal{T}_\alpha \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} x = \dfrac{x_0\,y_0}{y_0 - \alpha} \\ y = y_0 - \alpha \end{cases}.$$

We verify this is an abelian group.

4)
$$\begin{cases} (1 + x^2)p - x\,y\,q = a, \\ \dfrac{p}{y} = b, \\ p\,\dfrac{x}{y} - q = c, \end{cases}$$

where a, b and c are constants. By eliminating p and q between the three relations, we obtain the representation of equilateral hyperbolas:

$$(cx + b)\, y = a.$$

Solution to Exercise 28

1) Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2\,r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2\,r^2\,\sin^2\,\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + f(r) + \frac{1}{r^2}\,g(\theta) + \frac{1}{r^2\,\sin^2\,\theta}\,h(\varphi) = 0,$$

justifying to write S in the form:

$$S = -E\,t + S_2(r) + S_3(\theta) + S_4(\varphi),$$

we obtain

$$S = -E\,t + \int \sqrt{\frac{2\alpha}{r^2} + 2E - 2\,f(r)}\,dr + \int \sqrt{\frac{2\beta}{\sin^2\,\theta} - 2\,g(\theta) - 2\,\alpha}\,d\theta + \int \sqrt{-2\,\beta - 2\,h(\varphi)}\,d\varphi,$$

where α and β are constants. The equations for time law and trajectories are, respectively

$$\begin{cases} \dfrac{\partial S}{\partial E} = -t + \displaystyle\int \frac{dr}{\sqrt{\dfrac{2\alpha}{r^2} + 2E - 2\,f(r)}} = t_0, \\[2ex] \dfrac{\partial S}{\partial \alpha} = \displaystyle\int \frac{dr}{r^2\sqrt{\dfrac{2\alpha}{r^2} + 2E - 2\,f(r)}} - \int \frac{d\theta}{\sqrt{\dfrac{2\beta}{\sin^2\,\theta} - 2\,g(\theta) - \,2\,\alpha}} = a, \\[2ex] \dfrac{\partial S}{\partial \beta} = \displaystyle\int \frac{d\theta}{\sin^2\,\theta\sqrt{\dfrac{2\beta}{\sin^2\,\theta} - 2\,g(\theta) - 2\,\alpha}} - \int \frac{d\varphi}{\sqrt{-2\,\beta - 2\,h(\varphi)}} = b, \end{cases}$$

where t_0, a and b are constants. Consequently,

$$\begin{cases} \left(\dfrac{dr}{dt}\right)^2 = 2\left(\dfrac{\alpha}{r^2} + E - f(r)\right), \\ \left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{2}{r^4}\left(\dfrac{\beta}{\sin^2\theta} - \alpha - g(\theta)\right), \\ \left(\dfrac{d\varphi}{dt}\right)^2 = \dfrac{2}{r^4\sin^4\theta}\left(-\beta - h(\varphi)\right). \end{cases}$$

2) Hamilton equations:

From the Hamiltonian

$$H = \frac{1}{2}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right] + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2\sin^2\theta},$$

we deduce

$$\begin{cases} \dot{r} = p_r, \quad \dot{\theta} = \dfrac{p_\theta}{r^2}, \quad \dot{\varphi} = \dfrac{1}{r^2\sin^2\theta}\,p_\varphi \\ \dot{p_r} = \dfrac{p_\theta^2}{r^3} + \dfrac{p_\varphi^2}{r^3\sin^2\theta} - f'(r) + \dfrac{2\,g(\theta)}{r^3} + \dfrac{2\,h(\varphi)}{r^3\sin^2\theta}, \\ \dot{p_\theta} = \dfrac{p_\varphi^2\cos\theta}{r^2\sin^3\theta} - \dfrac{g'(\theta)}{r^2} + \dfrac{2\,h(\varphi)}{r^2\sin^3\theta}\cos\theta, \quad \dot{p_\varphi} = -\dfrac{h'(\varphi)}{r^2\sin^2\theta}. \end{cases}$$

We can add $\dot{h} = 0$, which corresponds to the conservation of energy. Finally, we obtain

$$\begin{cases} r^4\dot{\varphi}^2\sin^2\theta = -2\beta - 2\,h(\varphi), \\ r^4\dot{\theta}^2 = 2\left[-\alpha + \dfrac{\beta}{\sin^2\theta} - g(\theta)\right], \\ \dot{r}^2 + 2\,f(r) - \dfrac{2\,\alpha}{r^2} = 2E. \end{cases}$$

Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\left(r^2\,\dot\varphi\,\sin^2\,\theta\right) = -\dfrac{h'(\varphi)}{r^2\,\sin^2\,\theta} \quad \Longrightarrow \quad \left(r^2\,\dot\varphi\,\sin^2\,\theta\right)^2 + 2\,h(\varphi) = -2\,\beta, \\ \dfrac{d}{dt}\left(r^2\,\dot\theta\right) = -\dfrac{g'(\theta)}{r^2} - \dfrac{2\,\beta\,\cos\,\theta}{r^2\,sin^3\,\theta} \quad \Longrightarrow \quad \left(r^2\,\dot\theta\right)^2 = 2\left[-\alpha + \dfrac{\beta}{\sin^2\,\theta} - g(\theta)\right], \end{cases}$$

to which the conservation of energy is added.

3) For $f = g = h = 0$, we obtain the equation of a straight line on which the motion is uniform.

Solution to Exercise 29

1) The Lagrange equation for y is

$$\frac{d}{dt}\left(\dot y\,\cos^2\,x\right) = 0 \quad \Longrightarrow \quad \dot y\,\cos^2\,x = \frac{\beta\,\omega}{2}.$$

The Lagrange equation for z is

$$\frac{d}{dt}\left(\dot z\,\sin^2\,x\right) = 0 \quad \Longrightarrow \quad \dot z\,\sin^2\,x = \frac{\gamma\,\omega}{2}.$$

Instead of the Lagrange equation for x, we use the conservation of energy:

$$2\,\dot x^2 + \dot y^2 \cos^2 x + \dot z^2 \sin^2 x + \frac{\omega^2}{2\,\sin^2 2x} = 2\,h,$$

where h is a constant. Taking into account the initial conditions and the first two integrals, we obtain

$$\begin{aligned} &2\,\dot x^2 + \frac{\omega^2}{4}\left(\frac{\beta^2}{\cos^2\,x} + \frac{\gamma^2}{\sin^2\,x} + \frac{2}{\sin^2\,2x}\right) \\ &\quad = \frac{\omega^2}{2}\,\left(1 + \beta^2 + \gamma^2 + 4\alpha^2\right) = k^2\frac{\omega^2}{2}, \end{aligned} \qquad [11.1]$$

with $k^2 = 1 + \beta^2 + \gamma^2 + 4\,\alpha^2$. Multiplying by $2\,\sin^2\,2x$ (for $2x \neq n\,\pi$) and denoting $u = \cos\,2x$, we obtain

$$\dot u^2 + k^2\,\omega^2\,u^2 + (\gamma^2 - \beta^2)\,\omega^2\,u - 4\,\alpha^2\,\omega = 0.$$

Deriving and simplifying by $2\,\dot{u}$, we obtain

$$\ddot{u} + k^2\,\omega^2\,u + \frac{\gamma^2 - \beta^2}{2}\,\omega^2 = 0$$

and by denoting $v = u - \dfrac{\beta^2 - \gamma^2}{2\,k^2}$, we obtain

$$\ddot{v} + k^2\,\omega^2\,v = 0.$$

Hence,

$$v = B\,\cos\,k\,\omega\,t + C\,\sin\,k\,\omega\,t$$

and

$$u = \cos\,2x = \frac{\beta^2 - \gamma^2}{2\,k^2} + B\,\cos\,k\,\omega\,t + C\,\sin\,k\,\omega\,t.$$

Taking into account the initial conditions, we obtain

$$\cos\,2x = \frac{\beta^2 - \gamma^2}{2\,k^2}\,(1 - \cos\,k\,\omega\,t) - \frac{2\alpha}{k}\,\sin\,k\,\omega\,t.$$

2) With respect to $\dot{x}$, from equation [11.1], we have $\dot{x}^2 = f(x)$, which can be represented in the form

$$\frac{4\,\dot{x}^2}{\omega^2} = k^2 - \frac{1 + 2\,\beta^2\,\sin^2\,x + 2\,\gamma^2\cos^2\,x}{\sin^2\,2x}.$$

The variations of $f(x)$ prove that when x goes to 0 or $\dfrac{\pi}{2}$, the right-hand side goes to $-\infty$, but at the initial instant, we have $x = \dfrac{\pi}{4}$ and $\dot{x}^2 = \alpha^2\,\omega^2 > 0$.

Thus, $\dot{x}$ is zero and changes sign at x_1, where $\dfrac{\pi}{4} > x_1 > 0$, and at x_2, where $\dfrac{\pi}{2} > x_2 > \dfrac{\pi}{4}$. Consequently, x belongs to $\left[0, \dfrac{\pi}{2}\right]$.

The parameter x remains constant if, at the initial instant, $\dot{x} = 0$ and $\ddot{x} = 0$. The Lagrange equation associated with x is

$$2\,\ddot{x} - \sin\,x\,\cos\,x\,\left(\dot{z}^2 - \dot{y}^2\right) = \frac{\omega^2\,\cos\,2x}{\sin^3\,2x}.$$

At the initial instant $x = \dfrac{\pi}{4}$ and $\dot{x}^2 = \alpha^2\,\omega^2$ and conditions $\dot{x}_0 = 0$, $x = \dfrac{\pi}{4}$ imply $\alpha = 0$ and $\sin 2x_0 = 1$, $\cos 2x_0 = 0$. Furthermore, from the differential equation obtained in (1) connecting u and $\ddot{u}$, where $u = \cos 2x$, the condition $\ddot{x}_0 = 0$ implies $\frac{1}{2}\left(\gamma^2 - \beta^2\right)\,\omega^2 = 0$ and $\gamma = \pm\beta$. Finally, x remains constant for $\alpha = 0$ and $\gamma = \pm\beta$.

3) If the constraint $\dot{y} + \dot{z} = \varphi(x)$, then we obtain the Lagrange equations

$$\frac{d}{dt}\left(\dot{y}\,\cos^2 x\right) = \lambda, \quad \frac{d}{dt}\left(\dot{z}\,\sin^2 x\right) = \lambda,$$

where λ is a Lagrange multiplier. We deduce

$$\frac{d}{dt}\left(\dot{y}\,\cos^2 x - \dot{z}\,\sin^2 x\right) = 0 \quad \Longrightarrow \quad \dot{y}\,\cos^2 x - \dot{z}\,\sin^2 x = D,$$

where D is a constant. We note that the constraint is not independent of time. Taking into account the constraint $\dot{y}\,\cos^2 x - [\varphi(x) - \dot{y}]\,\sin^2 x = D$, we deduce

$$\dot{y} = D + \varphi(x)\,\sin^2 x \quad \text{and similarly} \quad \dot{z} = -D + \varphi(x)\,\cos^2 x.$$

For x, the Lagrange equation is

$$2\,\ddot{x} - \sin x\,\cos x\,\left[\varphi(x)^2\left(\cos^2 x - \sin^2 x - 2\,D\,\varphi(x)\right)\right] - \frac{\omega^2\,\cos 2x}{\sin^3 2x} = 0$$

which yields a differential equation in the form $\dot{x}^2 = F(x)$ by integration.

If $\varphi(x) = 0$, then the constraint is independent of time. From

$$\dot{y} = D = \beta\,\omega \quad \text{and} \quad \dot{z} = -D = -\beta\,\omega,$$

we deduce

$$y = \beta\,\omega\,t + y_0 \quad \text{and} \quad z = -\beta\,\omega\,t + z_0.$$

Hence, the conservation of energy can be written as

$$4\,\dot{x}^2 + \beta^2\,\omega^2 + \frac{\omega^2}{2\,\sin^2 2x} = \left(2\,\alpha^2 + \beta^2 + \frac{1}{2}\right)\,\omega^2$$

or

$$4\,\dot{x}^2\,\sin^2 2x - \omega^2\,\left(1 + 4\,\alpha^2\right)\,\sin^2 2x + \omega^2 = 0.$$

Let us denote $u = \cos\, 2x$; then, $2\,\dot{x}\,\sin\, 2x = -\dot{u}$ and $\sin^2\, 2x = 1 - u^2$. Hence,

$$\dot{u}^2 + \omega^2\,\left(1 + 4\,\alpha^2\right)\,u^2 - 4\,\alpha^2\,\omega^2 = 0 \quad \Longrightarrow \quad \ddot{u} + \omega^2\,\left(1 + 4\,\alpha^2\right)\,u = 0$$

and $u = B\,\cos\, k\,\omega\,t + C\,\sin\, k\,\omega\,t$ with $k^2 = 1 + 4\,\alpha^2$.

For $t = 0$, $x = \dfrac{\pi}{4}$, $\dot{x} = \alpha\,\omega$,

$$\cos\, 2x = 0 = B \quad \text{and} \quad 2\,\dot{x}\,\sin\, 2x = 2\,\dot{x} = 2\,\alpha\,\omega = -C\,k\,\omega.$$

Therefore, $C = -\dfrac{2\,\alpha}{k}$. Finally,

$$\cos\, 2x = -\frac{2\,\alpha}{k}\,\sin\, k\,\omega\,t.$$

Solution to Exercise 30

For a canonical change of variables, we must have $\alpha^2 - \beta^2 = 1$. The generating function is $F(q,s) = -\dfrac{1}{2\beta}(q-s)^2$.

Solution to Exercise 31

1) The change of variables is canonical if and only if $[r,s]_{p,q} = 1$. We verify that $\dfrac{\partial r}{\partial p}\dfrac{\partial s}{\partial q} - \dfrac{\partial r}{\partial q}\dfrac{\partial s}{\partial p} = 1$.

2) The generating function S satisfies $dS = p\,dq - r\,ds$. Furthermore, $d(S - p\,q) = -r\,ds - q\,dp$ and $F(s,p) = S - pq$, with

$$\begin{cases} \dfrac{\partial F}{\partial s} = -r = -e^{-s}\,\cos\, p \\ \dfrac{\partial F}{\partial p} = -q = -e^{-s}\,\sin\, p \end{cases}.$$

We obtain $F = e^{-s}\,\cos\, p + Cte$. Then, $S = e^{-s}\,\cos\,[\mathrm{Arcsin}\,(q\,e^s)] + q\,\mathrm{Arcsin}\,(q\,e^s)]$.

3) We have $G = S + r\,s + Cte = F + p\,q + r\,s + Cte$. Hence,

$$G(q,r) = r + q\,\mathrm{Arctg}\,(\frac{q}{r}) - r\,\mathrm{Ln}\,\sqrt{q^2 + r^2} + Cte.$$

Solution to Exercise 32

Using calculations similar to those used in Exercise 31, we obtain

1) $\alpha = \dfrac{1}{2}$ and $\beta = 1$.

2) $S(q,s) = -\dfrac{s^2}{2}\operatorname{cotg} 2q + Cte, \quad F(r,q) = \dfrac{r^2}{2}\operatorname{tg} 2q + Cte,$

$$G(p,r) = -\frac{r}{2}\sqrt{p-r^2} - \frac{p}{2}\operatorname{Arccos}\left(\frac{r}{\sqrt{p}}\right) + Cte,$$

$$H(p,s) = -\frac{s}{2}\sqrt{p-r^2} + \frac{p}{2}\operatorname{Arcsin}\left(\frac{s}{\sqrt{p}}\right) + Cte.$$

Solution to Exercise 33

This is the converse problem:

$$dS = p\,dq - r\,ds + (K-H)dt.$$

S is independent of t, which implies that $K = H$. Therefore,

$$dS = p\,dq - r\,ds \quad \text{and} \quad p = \frac{\partial S}{\partial q},\ r = -\frac{\partial S}{\partial s}.$$

We deduce

$$p = \frac{s^2}{\sin^2 2q} \quad \text{and} \quad r = s\operatorname{cotg} 2q.$$

Finally,

$$r = \sqrt{p}\cos 2q \quad \text{and} \quad s = \sqrt{p}\sin 2q.$$

Solution to Exercise 34

The calculations of 1) and 2) are similar to those in Exercises 32 and 33. As a result, we get:

$$F(p,s) = -(e^s - 1)^2 \operatorname{tg} p + Cte, \quad F \text{ is obtained by writing} \quad S = F + p\,q.$$

Solution to Exercise 35

1) The elliptical coordinate $q_1 = Cte$ corresponds to the ellipses of foci F and F', center O and half-axes $a = \operatorname{ch} q_1,\ b = |\operatorname{sh} q_1|$. The foci are at a distance 1 from O. The coordinate $q_2 = Cte$ corresponds to the hyperbolas of foci F and F'. According to the Poncelet theorem, the ellipses and hyperbolas are orthogonal.

2) Potentials associated with F and F' are $W_F = -\dfrac{\mu}{r}$ and $W_{F'} = -\dfrac{\mu'}{r'}$, respectively, where r is the distance from M to F and r' from M to F'. We deduce

$$2\,T(q_1, q_2, \dot{q}_1, \dot{q}_2) = \left[\operatorname{sh}^2 q_1 + \sin^2 q_2\right]\,\left(\dot{q}_1^2 + \dot{q}_2^2\right),$$

$$W(q_1, q_2) = -\frac{(\mu + \mu')\operatorname{ch} q_1 + (\mu - \mu')\,\cos\, q_2}{\operatorname{sh}^2 q_1 + \sin^2 q_2}.$$

3) Consequently,

$$T = (U_1 + U_2)\,(V_1\,\dot{q}_1 + V_2\,\dot{q}_2) \quad \text{with} \quad U_1 = \sin^2\, q_1,$$
$$V_1 = 1,\ U_2 = \sin^2\, q_2,\ V_2 = 1$$

and

$$W = \frac{W_1 + W_2}{U_1 + U_2} \quad \text{with} \quad W_1 = -(\mu + \mu')\operatorname{ch} q_1,\ W_2 = -(\mu - \mu')\,\cos\, q_2.$$

We will now consider Liouville's integrability, with

$$p_1 = \left[\operatorname{sh}^2 q_1 + \sin^2\, q_2\right]\,\dot{q}_1, \quad p_2 = \left[\operatorname{sh}^2 q_1 + \sin^2\, q_2\right]\,\dot{q}_2$$

and

$$2\,H = \frac{p_1^2 + p_2^2}{\operatorname{sh}^2 q_1 + \sin^2\, q_2} - \frac{2\,\alpha \operatorname{ch} q_1 + 2\,\beta\,\cos\, q_2}{\operatorname{sh}^2 q_1 + \sin^2\, q_2} \quad \text{where}$$
$$\alpha = \mu + \mu' \text{ and } \beta = \mu - \mu'.$$

Let us denote $S = -E\,t + S_1(q_1) + S_2(q_2)$. The Jacobi equation is $H + \dfrac{\partial S}{\partial t} = 0$ and

$$S'^2_1 + S'^2_2 - 2\,\alpha \operatorname{ch} q_1 - 2\,\beta\,\cos\, q_2 = 2\,E\,\left[\operatorname{sh}^2 q_1 + \sin^2\, q_2\right],$$

which can be separated into two equations:

$$\begin{cases} S'^2_1 = 2\,\alpha\,\mathrm{ch}\,q_1 + 2\,E\,\mathrm{sh}^2\,q_1 + k, \quad \text{with} \quad S_1 = S_1(q_1, E, k,), \\ S'^2_2 = 2\,\beta\cos q_2 + 2\,E\sin^2 q_2 - k, \quad \text{with} \quad S_2 = S_2(q_2, E, k), \end{cases}$$

where k is a constant. We obtain S_1 and S_2, and deduce

$$\begin{cases} \dfrac{\partial S}{\partial E} = Cte \quad \Longrightarrow \quad t - t_0 = \dfrac{\partial S_1}{\partial E} + \dfrac{\partial S_2}{\partial E}, \quad \text{the time law,} \\ \dfrac{\partial S}{\partial k} = Cte \quad \Longrightarrow \quad \dfrac{\partial S_1}{\partial k} + \dfrac{\partial S_2}{\partial k} = 0, \text{ which gives the equation of the trajectory.} \end{cases}$$

4) In the case of the conservation of energy, we obtain

$$\boldsymbol{P}^\star\,d\boldsymbol{Q} - \boldsymbol{R}^\star\,d\boldsymbol{S} + (K - H)\,dt = \frac{\partial F}{\partial t}\,dt + \frac{\partial F}{\partial \boldsymbol{Q}}\,d\boldsymbol{Q} + \frac{\partial F}{\partial \boldsymbol{R}}\,d\boldsymbol{R} - d(\boldsymbol{R}^\star\,\boldsymbol{S}),$$

hence

$$\frac{\partial F}{\partial \boldsymbol{Q}} = \boldsymbol{P}^\star, \quad \frac{\partial F}{\partial \boldsymbol{R}} = \boldsymbol{S}^\star, \quad K - H = \frac{\partial F}{\partial t} + 0.$$

The new Hamilton equations are

$$\dot{\boldsymbol{R}}^\star = -\frac{\partial K}{\partial \boldsymbol{S}} = \boldsymbol{0}, \text{ then} \quad \boldsymbol{R} = \boldsymbol{R}_0 \quad \text{and} \quad \dot{\boldsymbol{S}}^\star = \frac{\partial K}{\partial \boldsymbol{R}}_{|\boldsymbol{R}=\boldsymbol{R}_0}$$
$$= \dot{\boldsymbol{S}}^\star{}_0, \text{ then} \quad \boldsymbol{S} = \dot{\boldsymbol{S}}_0\,t + \boldsymbol{S}_0.$$

Consequently, $K(\boldsymbol{R}_0) = H = E = \beta_1$. So, the Jacobi equation can be written as

$$H\left[\left(\frac{\partial F}{\partial \boldsymbol{Q}}\right)^\star, \boldsymbol{Q}\right] = \beta_1.$$

A solution includes β_1 and $n-1$ non-additive parameters $\beta_2, \ldots, \beta_n$, i.e. $F(\boldsymbol{Q}, \boldsymbol{B})$, where $\boldsymbol{B} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \boldsymbol{B}_0$. Since $\boldsymbol{B} = \boldsymbol{B}_0$ and $\boldsymbol{R} = \boldsymbol{R}_0$, we can write $\boldsymbol{B} = G(\boldsymbol{R}_0)$, with $G = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$ and the solution giving $F(\boldsymbol{Q}, G(\boldsymbol{R}_0))$. The equations

of motion are $\boldsymbol{S} = \left(\dfrac{\partial F}{\partial \boldsymbol{R}_0}\right)^{\star}$. Generally, we choose $G = \mathbf{1}$, where $\mathbf{1}$ is the identity and we obtain the equations of motion of the type obtained in (4).

5) Refer to the book.

Solution to Exercise 36

A. 1) $2\,T = m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right)$ and $W = -\dfrac{k}{r}$. The conjugate variables and the Hamiltonian are:

$$p_r = m\dot{r}, \quad p_\theta = m\,r^2\,\dot{\theta} \quad \text{and} \quad H = \frac{1}{2\,m}\left(p_r^2 + \frac{1}{r^2}\,p_\theta^2\right) - \frac{k}{r}.$$

2) Since t is a secondary parameter, we can write S in the form $S = -E\,t + F(\theta, r)$. We obtain the Jacobi equation:

$$\frac{1}{2\,m}\left[\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}\right)^2\right] - \frac{k}{r} = E$$

or

$$r^2\left(\frac{\partial F}{\partial r}\right)^2 - 2\,m\,k\,r - 2\,m\,E\,r^2 + \left(\frac{\partial F}{\partial \theta}\right)^2 = 0,$$

which allows us to write $F(\theta, r) = S_1(r) + S_2(\theta)$ and yields

$$\left(\frac{\partial S_1}{\partial r}\right)^2 = 2\,m\left[E + \frac{k}{r} - \frac{\alpha^2}{r^2}\right], \quad S_2 = \alpha\sqrt{2\,m}\,\theta.$$

Hence,

$$S = -E\,t + \sqrt{2\,m}\left[\alpha\,\theta + \varepsilon\int\sqrt{E + \frac{k}{r} - \frac{\alpha^2}{r^2}}\right]\,dr,$$

where α is a constant and $\varepsilon = \pm 1$. This leads to the time law:

$$\frac{\partial S}{\partial E} = \varepsilon\sqrt{2\,m}\int\frac{\sqrt{2\,m}\,dr}{\sqrt{E + \dfrac{k}{r} - \dfrac{\alpha^2}{r^2}}} = t - t_0$$

and the equation of the trajectory:

$$\frac{\partial S}{\partial \alpha} = \sqrt{2\,m}\theta - \varepsilon\sqrt{2\,m}\,\alpha \int \frac{dr}{r^2\sqrt{E + \dfrac{k}{r} - \dfrac{\alpha^2}{r^2}}} = \alpha_0,$$

where t_0 and α_0 are constants. The law of areas is $p_\theta = Cte$ or $r^2\,\dot{\theta} = C$.

3) If the trajectory is closed, then the motion is periodic. We use the action variables with $r_1 \leq r \leq r_2$:

$$J_r = \oint \varepsilon\sqrt{2\,m}\sqrt{E + \frac{k}{r} - \frac{\alpha^2}{r^2}}\,dr \quad \text{and} \quad J_\theta = \oint \alpha\sqrt{2\,m}\,d\theta = 2\,\pi\,\alpha\sqrt{2\,m},$$

where the integral denoted by $\oint$ is taken over a period of the motion.

4) We denote $f(z) = \sqrt{E + \dfrac{k}{z} - \dfrac{\alpha^2}{z^2}}$; r_1 and r_2 are the roots of the polynomial $E\,r^2 + k\,r - \alpha^2$ written as $E\,(r - r_2)\,(r - r_1)$. Since $r_1 \leq r \leq r_2$, we must have $E < 0$ and

$$\int_{r_1}^{r_2} \sqrt{E(r - r_1)(r - r_2)}\,\frac{dr}{r} = \int_{r_1}^{r_2} f(r)\,dr.$$

5) We have $r_1\,r_2 = -\dfrac{\alpha^2}{E} > 0$ and $r_1 + r_2 = -\dfrac{k}{E} > 0$; hence, $0 < r_1 < r_2$. Consider the plane shown in Figure 11.1. The domain located between Γ, Γ_1 and Γ_2 is denoted by D. Then,

$$\int_\Gamma f(r)\,dr + \int_{\Gamma_1} f(r)\,dr = \int_{\Gamma_2} f(r)\,dr, \quad \text{which implies} \quad \int_\Gamma f(r)dr = \frac{J_r}{\sqrt{2\,m}}.$$

We have $\displaystyle\int_{\Gamma_1} f(r)\,dr = 2\,i\pi R_0$, where R_0 is the residue associated with the single pole $z = 0$. $R_0 = \lim_{z\to 0} z\,f(z) = -i\,\alpha$ because with the same determination for $f(z)$, we have $\arg f(z) = \dfrac{3\,\pi}{2}$. $\displaystyle\int_\Gamma f(r)\,dr = \frac{J_r}{\sqrt{2\,m}} = -2\,i\pi\,(R_0 + R_\infty)$. Let us calculate R_∞: by the change of variable $u = \dfrac{1}{z}$, we have $f(z)\,dz = g(u)\,du$, where

$g(u) = \dfrac{\sqrt{-E}}{u^2}\left(1 + \dfrac{k}{E}\,u - \dfrac{\alpha^2}{E}\,u^2\right)^{\frac{1}{2}}$. Hence, $R_\infty = \dfrac{i\,k}{2\sqrt{-E}}$, and finally,

$$J_r = -2\,\pi\sqrt{2\,m}\,\alpha + \frac{\pi\,k}{\sqrt{-E}}\sqrt{2\,m}.$$

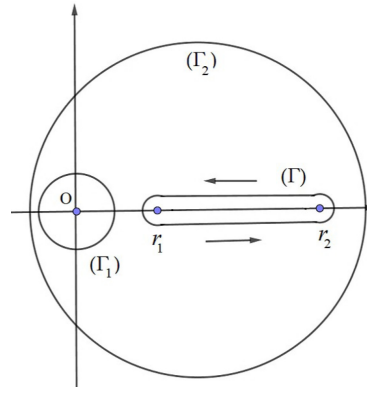


Figure 11.1. *Calculation of J_r using the residue method*

Consequently,

$$J_r + J_\theta = \frac{k\,\pi}{\sqrt{-E}}\sqrt{2\,m} \quad \text{then} \quad \frac{2\,m\,\pi^2\,k^2}{(J_r + J_\theta)^2} = -E.$$

6) The frequencies of the motion are:

$$\nu_r = \nu_\theta = \frac{\partial E}{\partial J_r} = \frac{\partial E}{\partial J_\theta} = \frac{4\,m\,\pi^2\,k^2}{(J_r + J_\theta)^3}$$

and the common period of the motion is

$$\tau_{\text{period}} = \frac{1}{\nu_r} = \frac{1}{\nu_\theta} = \pi\,k\sqrt{\frac{1}{-2\,E^3}}.$$

B. In the three-dimensional space and spherical representation,

$$2\,T = m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\,\theta\,\dot{\varphi}^2\right),\ W = -\frac{k}{r}.$$

The conjugate variables and the Hamiltonian are:

$$p_r = m\,\dot{r}, \quad p_\theta = m\,r^2\,\dot{\theta}, \quad p_\varphi = m\,r^2\,\sin^2\,\theta\,\dot{\varphi},$$

and

$$H = \frac{1}{2\,m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\,\sin^2\,\theta}\right) - \frac{k}{r}.$$

We deduce the Jacobi equation:

$$\frac{1}{2\,m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\,\sin^2\,\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2\right] - \frac{k}{r} = E,$$

with

$$S = -E\,t + \sqrt{2\,m}\int\sqrt{E + \frac{k}{r} - \frac{\alpha^2}{r^2}}\,dr + \sqrt{2\,m}\int\sqrt{\alpha^2 - \frac{\beta^2}{\sin^2\,\theta}}d\theta + \sqrt{2\,m}\,\beta\,\varphi,$$

where α and β are constants. We obtain the action variables

$$\begin{cases} J_\varphi = \oint p_\varphi d\varphi = \sqrt{2\,m}\oint \beta\,d\varphi = 2\,\sqrt{2\,m}\,\pi\,\beta, \\ \\ J_\theta = \oint p_\theta\,d\theta = \sqrt{2\,m}\oint\sqrt{\alpha^2 - \dfrac{\beta^2}{\sin^2\,\theta}}\,d\theta, \\ \\ J_r = \oint p_r\,dr = \sqrt{2\,m}\oint\sqrt{E + \dfrac{k}{r} - \dfrac{\alpha^2}{r^2}}\,dr = \sqrt{2\,m}\,2\,\pi\left(-\alpha + \dfrac{k}{2\,\sqrt{-E}}\right). \end{cases}$$

To calculate J_θ, we express the kinetic energy both in the spherical and planar polar representations:

$$\frac{2\,T}{m} = \dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\,\theta\,\dot{\varphi}^2 = \dot{r}^2 + r^2\,\dot{\psi}^2$$

where ψ is the azimuthal angle of the planet P on its orbit (angle denoted by θ in I). We deduce:

$$p_r\,\dot{r} + p_\psi\,\dot{\psi} = p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\varphi\,\dot{\varphi} \quad \Longrightarrow \quad p_\theta\,d\theta = p_\psi\,d\psi - p_\varphi\,d\varphi$$

and

$$J_\theta = \oint p_\psi\,d\psi - \oint p_\varphi\,d\varphi = \sqrt{2\,m}\,2\,\pi(\alpha - \beta).$$

Consequently,

$$\sqrt{2\,m}\,\alpha = \frac{1}{2\,\pi}\,(J_\theta + J_\varphi) \quad \Longrightarrow \quad J_r = \oint \sqrt{2\,m\,E + \frac{2\,m\,k}{r} - \frac{(J_\theta + J_\varphi)^2}{4\,\pi^2 r^2}}\,dr.$$

By a similar calculation of I,

$$J_r = -\,(J_\theta + J_\varphi) + \pi\,k\sqrt{\frac{2\,m}{-E}},$$

hence,

$$H \equiv E = -\frac{2\,\pi^2\,m\,k^2}{(J_r + J_\theta + J_\varphi)^2}$$

and the common frequency is

$$\nu = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\varphi} = \frac{4\,\pi^2\,m\,k^2}{(J_r + J_\theta + J_\varphi)^2}.$$

Solution to Exercise 37

1) The solutions of the system are:

$$\begin{cases} x_1 = \lambda\,e^{-t} + \mu\,e^{2t} \\ \\ x_2 = -\dfrac{\lambda}{2}\,e^{-t} + \mu\,e^{2t} \end{cases}$$

where λ and μ are constants depending on initial conditions. The equilibrium position corresponds to $x_1 = x_2 = 0$. The equilibrium position is Lyapunov unstable because when t goes to infinity, x_1 and x_2 go to infinity. By writing the differential system

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2), \\ \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2), \end{cases}$$

we obtain

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \\ 1 & 1 \end{bmatrix} \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

with $\mathrm{Tr}\left(\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) \equiv \mathrm{div}\,\boldsymbol{f}(\boldsymbol{x}) = 1$. The divergence of $\boldsymbol{f}(\boldsymbol{x})$ is not zero, and transformation g^t does not conserve the areas in the plane.

2) – At $t = 0$, $x_1(0) = \lambda + \mu$ and $x_2(0) = \mu - \dfrac{\lambda}{2}$. Therefore, $-1 \leq \lambda + \mu \leq 1$ and $-1 \leq \mu - \dfrac{\lambda}{2} \leq 1$.

– At $t = 1$,

$$x_1(1) = \frac{\lambda}{e} + \mu\, e^2,\ x_2(1) = \mu - \frac{\lambda}{2\,e} + \mu\, e^2, \quad \text{therefore}$$

$$\lambda = \frac{2\,e}{3}\,(x_1 - x_2),\ \mu = \frac{x_1 + 2\,x_2}{3\,e^2}.$$

We deduce the inequalities:

$$\begin{cases} -1 \leq \dfrac{x_1}{3}\left(\dfrac{1}{e^2} + 2\,e\right) + \dfrac{2\,x_2}{3}\left(\dfrac{1}{e^2} - e\right) \leq 1 \\ -1 \leq \dfrac{x_1}{3}\left(\dfrac{1}{e^2} - e\right) + \dfrac{x_2}{3}\left(\dfrac{2}{e^2} + e\right) \leq 1 \end{cases}$$

which represents in plane (x_1, x_2) the interior surface of the parallelogram delimited by four lines. It can be explicitly verified that g^t does not conserve areas.

Solution to Exercise 38

1) The equation can be written as

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -\sin x. \end{cases}$$

The equilibrium position is obtained for $x = 0$.

The study of linearized motions is obtained by considering $\boldsymbol{f}(x, y) = \begin{bmatrix} y \\ -\sin x \end{bmatrix}$ whose tangent linear application is

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}, \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We have $\mathrm{Tr}\left(\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right) = \mathrm{div}\,\boldsymbol{f} = 0$. The volume is conserved in the phase space.

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

whose eigenvalues are i and $-i$. The point $(0,0)$ is of stable equilibrium for the linearized system. However, since the real parts of the eigenvalues are zero, we cannot conclude that it is the same for the original system. Nonetheless, from a mechanical point of view, with the kinetic energy $T = \frac{1}{2}\,\dot{x}^2$ and the potential energy $W = -\cos x$, the Lejeune–Dirichlet theorem allows us to conclude on an equilibrium which is Lyapunov stable. Due to the conservation of energy, the equilibrium cannot be asymptotically stable (energy cannot go to zero when t goes to infinity).

2) The mechanical method concludes that there is Lyapunov stable equilibrium for $x = 0$; however, $x = 0$ is not asymptotically stable. The "volume" is conserved in the phase space.

Solution to Exercise 39

1) The equilibrium points correspond to $\sin x = 0$ and $\sin y = 0$, i.e. to $x = 0$ and $y = 0$ (modulo π).

2) We limit ourselves to $x = 0$, $y = 0$; the reader can study the other cases. Denoting $f(x,y) = -\sin y$ and $g(x,y) = \sin x + \sin y$, we obtain at $(0,0)$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{X}} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}_{(x=0,y=0)} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{where} \quad \boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \boldsymbol{F} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

3) With the near equilibrium position $x = 0, y = 0$, the linearized system is

$$\begin{cases} \dfrac{dx}{dt} = -y \\[2ex] \dfrac{dy}{dt} = x + y \end{cases}$$

which implies that $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + x = 0$ and $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} + y = 0$. The solutions for x or y are $\lambda\, e^{r_1\, t} + \mu\, e^{r_2\, t}$, where $r_j = \dfrac{1 \pm i\sqrt{3}}{2}$, $j \in \{1,2\}$. The solutions of the linearized equation are Lyapunov unstable since the real parts of r_1 and r_2 are positive.

Solution to Exercise 40

1) The kinetic energy is

$$T = \frac{1}{2}\, m\, \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\, \left[\dot{x}^2 + \dot{y}^2 + \left(\frac{\partial f}{\partial x}\,\dot{x} + \frac{\partial f}{\partial y}\,\dot{y}\right)^2\right]$$

The potential energy is

$$W = m\, g\, f(x,y) = g\, f(x,y).$$

2) The point O belongs to the surface and $f(0,0) = 0$. The Lejeune–Dirichlet theorem implies that O corresponds to a minimum of the potential W, and for $(x,y) \neq (0,0)$, we have $f(x,y) > 0$ near $(0,0)$.

3) In order for O to be an extremum point, it is necessary that

$$p(0,0) = \frac{\partial f}{\partial x}(0,0) = 0, \quad q(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

The Taylor–Lagrange development of $f(x,y)$ near $(0,0)$ is

$$f(x,y) - f(0,0) = x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + \frac{1}{2\,!}\left(x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0)\right)^{(2)}$$
$$+\frac{1}{3\,!}\left(x\frac{\partial f}{\partial x}(h,k) + y\frac{\partial f}{\partial y}(h,k)\right)^{(3)}, \quad \text{where } 0 < h < x \text{ and } 0 < k < y.$$

In this case,

$$f(x,y) = \frac{1}{2\,!}\left(x^2\frac{\partial^2 f}{\partial x^2}(0,0) + 2\,xy\frac{\partial^2 f}{\partial x \partial y}(0,0) + y^2\frac{\partial^2 f}{\partial y^2}(0,0)\right)$$
$$+\ \frac{1}{3\,!}\left(x\frac{\partial f}{\partial x}(h,k) + y\frac{\partial f}{\partial y}(h,k)\right)^{(3)} \quad \text{where } 0 < h < x \text{ and } 0 < k < y,$$

and at$(0,0)$,

$$f(x,y) - f(0,0) > 0 \quad \Longrightarrow \quad s^2 - r\,t < 0.$$

We denote $r(0,0) = r,\ s(0,0) = s,\ t(0,0) = t$; the linearization of the kinetic energy and the potential energy in the neighborhood of $(0,0)$ can be written as

$$T_\ell = \frac{1}{2}\,[\dot{x}, \dot{y}] \begin{bmatrix} 1+p^2 & p\,q \\ p\,q & 1+p^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

and

$$W_\ell = \frac{g}{2}\left(x^2 \frac{\partial^2 f}{\partial x^2}(0,0) + 2\,xy \frac{\partial^2 f}{\partial x \partial y}(0,0) + y^2 \frac{\partial^2 f}{\partial y^2}(0,0)\right) = \frac{g}{2}\,[x, y] \begin{bmatrix} r & s \\ s & t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We denote

$$\mathcal{A} = \begin{bmatrix} 1+p^2 & p\,q \\ p\,q & 1+p^2 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} r & s \\ s & t \end{bmatrix}.$$

The eigenfrequencies ω of the system are associated with the eigenvalues $\lambda = \omega^2$ of $\mathcal{B}$ with respect to $\mathcal{A}$ and whose characteristic polynomial is

$$\det\left(\mathcal{B} - \lambda\,\mathcal{A}\right) = \lambda^2 - \lambda\,g\,(r+t) + g^2\,\left(rt - s^2\right) = 0.$$

The sum of the roots of the polynomial is $g\,(r+t)$, and the product of roots is $g^2\,(rt - s^2)$. Since at $(0,0)$, $p = q = 0$,

$$\left(\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right) = r + t \quad \text{and} \quad \frac{1}{\mathrm{R}_1 \mathrm{R}_2} = rt - s^2,$$

and $\dfrac{g}{\mathrm{R}_1}$, $\dfrac{g}{\mathrm{R}_2}$ are the roots of $\det\left(\mathcal{B} - \lambda\,\mathcal{A}\right) = 0$. Since $s^2 - rt < 0$, the radii of curvature R_1 and R_2 have the same sign. We choose t and r to be positive; thus, $\dfrac{1}{\mathrm{R}_1}$ and $\dfrac{1}{\mathrm{R}_2}$ are positive, and we denote

$$\omega_1 = \sqrt{\frac{g}{\mathrm{R}_1}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{g}{\mathrm{R}_2}}$$

which are the eigenfrequencies. In the basis of eigenvectors, the system is written as:

$$\ddot{q}_1 + \omega_1^2\,q_1 = 0, \quad \ddot{q}_2 + \omega_2^2\,q_2 = 0,$$

where q_1 and q_2 are the coordinates in this basis. We obtain

$$q_1 = A\,\cos\left(\sqrt{\frac{g}{\mathrm{R}_1}}\,t + \varphi_1\right), \quad q_2 = B\,\cos\left(\sqrt{\frac{g}{\mathrm{R}_2}}\,t + \varphi_2\right).$$

where A, B, φ_1 and φ_2 are constants. A general motion is a linear combination of eigenmotions and the small motions near the stable equilibrium position O are written:

$$\begin{cases} x = a_1\,\cos\left(\sqrt{\dfrac{g}{\mathrm{R}_1}}\,t + \varphi_1\right) + b_1\,\cos\left(\sqrt{\dfrac{g}{\mathrm{R}_2}}\,t + \varphi_2\right), \\ x = a_2\,\cos\left(\sqrt{\dfrac{g}{\mathrm{R}_1}}\,t + \varphi_1\right) + b_2\,\cos\left(\sqrt{\dfrac{g}{\mathrm{R}_2}}\,t + \varphi_2\right), \end{cases}$$

where a_1, b_1, a_2 and b_2 are constants.

Solution to Exercise 41

1) With respect to the orthonormal coordinate system $O\,\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, the kinetic energy and potential energy are

$$T = \frac{1}{2}\,m\,R^2\left(\dot\theta^2 + \dot\psi^2\,\sin^2\theta\right), \quad W = m\,g\,R\cos\,\theta.$$

The Lagrange equations are

$$\frac{d}{dt}\left(m\,R^2\,\dot\psi\,\sin^2\,\theta\right) = 0 \quad \Longleftrightarrow \quad \dot\psi\,\sin^2\,\theta = k,$$

where k is a constant, and

$$\frac{d}{dt}\left(m\,R^2\,\dot\theta\right) - m\,R^2\,\dot\psi^2\,\sin\,\theta\,\cos\,\theta = m\,g\,R\,\sin\,\theta.$$

Taking into account the previous first integral, this can be written as:

$$\ddot\theta - k^2\frac{\sin\,\theta\,\cos\,\theta}{\sin^4\,\theta} = \frac{g}{R}\,\sin\,\theta,$$

and gives the equation of the conservation of energy.

2) In this case,

$$T = \frac{1}{2}\,m\,R^2\left(\dot\theta^2 + \omega^2\,\sin^2\theta\right), \quad W = m\,g\,R\cos\,\theta$$

and we obtain the single Lagrange equation:

$$\ddot{\theta} - \omega^2 \sin\theta \cos\theta = \frac{g}{R} \sin\theta.$$

3) Equilibrium positions $\theta = \theta_0$ satisfy $\dot{\theta}_0 = 0$ and $\ddot{\theta}_0 = 0$. We deduce

$$\theta_0 = 0 \text{ or } \pi \text{ (modulo } \pi) \quad \text{and} \quad \frac{g}{R} + \omega^2 \cos\theta_0 = 0.$$

a) If $\theta_0 = 0$, the linearized Lagrange equation is

$$\ddot{\theta} - \left(\omega^2 + \frac{g}{R}\right) \theta = 0.$$

The position $\theta_0 = 0$ is the unstable equilibrium position.

b) If $\theta_0 = \pi$, we can write $\alpha = \pi - \theta$ and the linearized Lagrange equation near $\alpha = 0$ is

$$\ddot{\alpha} + \left(\frac{g}{R} - \omega^2\right) \alpha = 0.$$

If $\dfrac{g}{R} > \omega^2$, then the equilibrium position is stable and the small motions are periodic with period $\tau = \dfrac{2\pi}{\sqrt{\dfrac{g}{R} - \omega^2}}$. If $\dfrac{g}{R} < \omega^2$, then the equilibrium position is unstable.

If $\dfrac{g}{R} = \omega^2$, then $\ddot{\alpha} = 0$ and we obtain $\alpha = \dot{\alpha}_0\, t + \alpha_0$. However, we can prove that the equilibrium position is stable.

c) If $\sin\theta_0 \neq 0$, then $\dfrac{g}{R} + \omega^2 \cos\theta_0 = 0$; therefore, it is necessary that $\dfrac{g}{R} < \omega^2$. By writing $\alpha = \theta - \theta_0$, we obtain the linearized Lagrange equation:

$$\ddot{\alpha} + \left(\omega^2 \sin^2\theta_0\right) \alpha = 0.$$

The equilibrium is stable and the period of the small motions is $\tau = \dfrac{2\pi}{\omega \sin\theta_0}$.

Solution to Exercise 42

1) The kinetic energy is

$$2T = 2m\ell^2 \dot{u}^2 + m\ell^2 \dot{v}^2 + 2m\ell^2 \dot{u}\dot{v} \cos(v - u).$$

2) The potential energy is

$$W = -m\,g\,\ell\,(2\,\cos\,u + \cos\,v).$$

We have

$$\frac{\partial^2 W}{\partial u^2} = 2\,m\,g\,\ell\,\cos\,u, \quad \frac{\partial^2 W}{\partial u \partial v} = 0, \quad \frac{\partial^2 W}{\partial v^2} = m\,g\,\ell\,\cos\,v.$$

The minimum of the potential energy is obtained for $u = 0$ and $v = 0$. According to the Lejeune–Dirichlet theorem, the corresponding position is a stable equilibrium.

3) a) The linearized kinetic energy T_ℓ and linearized potential energy W_ℓ near the position $(u = \alpha = 0, v = \beta = 0)$ are

$$2\,T_\ell = 2\,m\ell^2\,\dot{\alpha}^2 + m\ell^2\,\dot{\beta}^2 + 2\,m\ell^2\,\dot{\alpha}\,\dot{\beta}, \quad W_\ell = m\,g\,\ell\,\left(\alpha^2 + \frac{\beta^2}{2}\right) + Cte.$$

b) We deduce the linearized equations of motion

$$2\,\ddot{\alpha} + \ddot{\beta} + 2\,\frac{g}{\ell}\,\alpha = 0, \quad \ddot{\alpha} + \ddot{\beta} + \frac{g}{\ell}\,\beta = 0,$$

c) The small motions in the form

$$\alpha = A\,\cos\,\omega\,(t - \tau), \quad \beta = B\,\cos\,\omega\,(t - \tau),$$

where τ is a constant, must be solutions of the differential system. We obtain

$$2\,A\,\left(\omega^2 - \frac{g}{\ell}\right) + B\,\omega^2 = 0, \quad A\,\omega^2 + B\,\left(\omega^2 - \frac{g}{\ell}\right) = 0.$$

The system of linear equations in A and B must have solutions other than the null solution, and we obtain

$$\omega^4 - 2\,\left(\omega^2 - \frac{g}{\ell}\right)^2 = 0,$$

which has the roots

$$\omega_1 = \pm\sqrt{\frac{\sqrt{2}}{\sqrt{2}-1}}\,\sqrt{\frac{g}{\ell}} \quad \text{with} \quad B = -\sqrt{2}\,A$$

and

$$\omega_2 = \pm\sqrt{\frac{\sqrt{2}}{\sqrt{2}+1}}\sqrt{\frac{g}{\ell}} \quad \text{with} \quad B = \sqrt{2}\,A.$$

The most general motion of the linearized system is

$$\alpha = a_1\,\cos\,\omega_1\,(t-\tau_1) + a_2\,\cos\,\omega_2\,(t-\tau_2),$$

$$\beta = -a_1\,\sqrt{2}\,\cos\,\omega_1\,(t-\tau_1) + a_2\,\sqrt{2}\,\cos\,\omega_2\,(t-\tau_2),$$

with a_1 and a_2 being constants.

Solution to Exercise 43

1) The kinetic energy of the triatomic molecule is

$$T = \frac{1}{2}\left(m\,\dot{\eta}_1^2 + M\,\dot{\eta}_2^2 + m\,\dot{\eta}_3^2\right) = \frac{1}{2}\left[\dot{\eta}_1, \dot{\eta}_2, \dot{\eta}_3\right]\,\mathcal{A}\,\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{\eta}_3\end{bmatrix},$$

where

$$\mathcal{A} = \begin{bmatrix} m & 0 & 0\\ 0 & M & 0\\ 0 & 0 & m\end{bmatrix}.$$

The potential energy of the triatomic molecule is

$$W = \frac{k}{2}\left[(\eta_2-\eta_1)^2 + (\eta_3-\eta_2)^2\right] = \frac{1}{2}\left[\eta_1, \eta_2, \eta_3\right]\,\mathcal{B}\,\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix},$$

where

$$\mathcal{B} = \begin{bmatrix} k & -k & 0\\ -k & 2\,k & -k\\ 0 & -k & k\end{bmatrix}.$$

2) The eigenfrequencies are the roots of $\det(\mathcal{B} - \lambda\,\mathcal{A}) = 0$:

$$\det\begin{bmatrix} k-\lambda\,m & -k & 0\\ -k & 2\,k-\lambda\,M & -k\\ 0 & -k & k-\lambda\,m\end{bmatrix} = (k-\lambda\,m)\,\left[m\,M\,\lambda^2 - \lambda\,k(M+2m)\right] = 0.$$

We obtain $\lambda_1 = \dfrac{k}{m}$, $\lambda_2 = 0$ and $\lambda_3 = \dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)$. The roots λ_1 and λ_3 are positive and give the eigenfrequencies $\omega_1 = \sqrt{\dfrac{k}{m}}$ and $\omega_3 = \sqrt{\dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)}$.

The eigenvalue $\lambda_2 = 0$ does not correspond to an oscillatory motion. The equation of motion for the corresponding normal coordinate is $\dot{\beta} = 0$ associated with a rectilinear and uniform translation. The zero frequency corresponds to the fact that the molecule undergoes a *rigid* translation along its axis, without any change in the potential energy; since the recall force is zero, the associated *frequency* must be null. We hypothesized that the molecule has three degrees of freedom, but one of them is the degree of freedom of the *rigid* body.

3) a) If the center of mass of the molecule is fixed, then $m\,(x_1 + x_3) + M\,x_2 = 0$.

b) From relation (3), we obtain a system with only two degrees of freedom, and

$$T = \frac{1}{2}\,[\dot{\eta}_1, \dot{\eta}_3]\,\mathcal{A}'\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_3\end{bmatrix} \quad \text{where} \quad \mathcal{A}' = \begin{bmatrix} m\left(1 + \dfrac{m}{M}\right) & \dfrac{m^2}{M} \\ \dfrac{m^2}{M} & m\left(1 + \dfrac{m}{M}\right)\end{bmatrix}.$$

Taking into account $m\,(x_{01} + x_{03}) + M\,x_{02} = 0$,

$$W = \frac{1}{2}\,[\eta_1, \eta_3]\,\mathcal{B}'\begin{bmatrix}\eta_1\\ \eta_3\end{bmatrix},$$

where

$$\mathcal{B}' = \begin{bmatrix} k\left[\left(1 + \dfrac{m}{M}\right)^2 + \left(\dfrac{m}{M}\right)^2\right] & 2\,k\,\dfrac{m}{M}\left(1 + \dfrac{m}{M}\right) \\ 2\,k\,\dfrac{m}{M}\left(1 + \dfrac{m}{M}\right) & k\left[\left(1 + \dfrac{m}{M}\right)^2 + \left(\dfrac{m}{M}\right)^2\right]\end{bmatrix}.$$

The roots of the characteristic polynomial $\det(\mathcal{B}' - \lambda\,\mathcal{A}') = 0$ are $\lambda'_1 = \dfrac{k}{m}$ and $\lambda'_3 = \dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)$. By fixing the mass center of the molecule, zero frequency was eliminated by preventing the molecule from moving as a whole along its axis.

4) Normal modes:

For $\omega_2 = 0$ and the normal coordinates (α, β, γ), the system is

$$\begin{cases} k\,\alpha - k\,\beta = 0 \\ -k\,\alpha + 2\,k\,\beta - k\,\gamma = 0 \\ -k\,\beta + k\,\gamma = 0 \end{cases} \quad \Longrightarrow \quad \alpha = \beta = \gamma.$$

Hence, the motion corresponds to a solid displacement from the position x_{01}, x_{02}, x_{03}.

For $\omega_1 = \sqrt{\dfrac{k}{m}}$, the system is written as:

$$\begin{cases} k\,\beta = 0 \\ -k\,\alpha + k\,\beta - k\,\gamma = 0 \end{cases} \quad \Longrightarrow \quad \beta = 0 \text{ and } \alpha = -\gamma.$$

The atom B does not move and the other atoms A and C vibrate out of phase from π.

For $\omega_3 = \sqrt{\dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)}$, the system is written as:

$$\begin{cases} \left[k - \sqrt{\dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)}\right] \alpha - k\,\beta = 0 \\ k\,\beta - \left[k - \sqrt{\dfrac{k}{m}\left(1 + \dfrac{2\,m}{M}\right)}\right] \gamma = 0 \end{cases} \quad \Longrightarrow \quad \alpha = \gamma.$$

The two atoms A and C vibrate with the same amplitude, while the central atom B oscillates, out of phase from π with respect to A and C, and with a different amplitude.

Solution to Exercise 44

1) The kinetic energy is $T = \dfrac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$.

2) The potential energy is

$$W = -(\cos\,\theta_1 + \cos\,\theta_2) + k\,\frac{a^2}{2},$$

where

$$a = \left[\sqrt{(\cos\,\theta_2 - \cos\,\theta_1)^2 + (1 + \sin\,\theta_2 - \sin\,\theta_1)^2} - 1 \right]$$

is the spring elongation. The linearized kinetic energy near $\theta_1 = \theta_2 = 0$ is

$$T_\ell = \frac{1}{2}\,[\dot\theta_1, \dot\theta_2]\,\mathcal{A}\,\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} \quad \text{where} \quad \mathcal{A} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

The linearized potential energy is

$$W_\ell = -\cos\,\theta_1 - \cos\,\theta_2 + \frac{k}{2}\,(\sin\,\theta_2 - \sin\,\theta_1)^2$$

with

$$\frac{\partial W_\ell}{\partial \theta_1} = \sin\,\theta_1 - k\,\cos\,\theta_1\,(\sin\,\theta_2 - \sin\,\theta_1)\,,$$

and

$$\frac{\partial W_\ell}{\partial \theta_2} = \sin\,\theta_2 + k\,\cos\,\theta_2\,(\sin\,\theta_2 - \sin\,\theta_1)\,.$$

These partial derivatives are zero for $\theta_1 = 0$ and $\theta_2 = 0$ corresponding to the equilibrium position of the system. The second derivatives of W_ℓ for $\theta_1 = 0$ and $\theta_2 = 0$ are

$$\frac{\partial^2 W_\ell}{\partial \theta_1^2} = 1 + k, \quad \frac{\partial^2 W_\ell}{\partial \theta_1 \partial \theta_2} = -k, \quad \frac{\partial^2 W_\ell}{\partial \theta_2^2} = 1 + k.$$

Hence,

$$W_\ell = \frac{1}{2}\,[\theta_1, \theta_2]\,\mathcal{B}\,\begin{bmatrix}\theta_1\\ \theta_3\end{bmatrix} \quad \text{where} \quad \mathcal{B} = \begin{bmatrix}1+k & -k\\ -k & 1+k\end{bmatrix}.$$

3) The eigenvalues of $\mathcal{B}$ with respect to $\mathcal{A}$ satisfy:

$$\det\begin{bmatrix}1 + k - \lambda & -k\\ -k & 1 + k - \lambda\end{bmatrix} = (1 - \lambda)\,(1 + 2\,k - \lambda) = 0,$$

which gives

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1 + 2\,k.$$

We obtain two unit eigenvectors:

$$\begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} \end{bmatrix}.$$

In the basis of unit eigenvectors, we denote

$$\theta_1 = \frac{1}{\sqrt{2}}\,(q_1 + q_2) \quad \text{and} \quad \theta_2 = \frac{1}{\sqrt{2}}\,(q_1 - q_2)$$

and we obtain

$$T_\ell = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) \quad \text{and} \quad W_\ell = \frac{1}{2}\left(q_1^2 + (1 + 2\,k)\,q_2^2\right).$$

The eigenfrequencies satisfy $\omega_i^2 = \lambda_i$, $i \in \{1, 2\}$; hence, $\omega_1 = 1$ and $\omega_2 = \sqrt{1 + 2\,k}$. By using the new coordinates, the Lagrange equations are

$$\ddot{q}_1 + q_1 = 0 \quad \text{and} \quad \ddot{q}_2 + (1 + 2\,k)\,q_2 = 0,$$

whose solutions, back to the initial space, are

$$\theta_1 = A\,\cos(t - \tau) + B\,\cos\sqrt{1 + 2\,k}\,(t - \tau),$$
$$\theta_2 = A\,\cos(t - \varphi) - B\,\cos\sqrt{1 + 2\,k}\,(t - \varphi),$$

where τ and φ are constants.

4) For $\theta_1 = \theta_2$, i.e. following the direction associated with $\omega_1 = 1$, we obtain $q_2 = 0$. The two pendulums are displaced with the eigenfrequencies 1; the spring is not required.

For $\theta_1 = -\theta_2$, we obtain $q_1 = 0$. The pendulums are in phase opposition along the direction associated with $\omega_2 = \sqrt{1 + 2\,k}$. The action of the spring increases the frequency.

5) At the initial instant $t = 0$, it is assumed that $\theta_1 = \theta_2 = 0$ and $\dot{\theta}_1 = v$, $\dot{\theta}_2 = 0$. In space (q_1, q_2), at $t = 0$, we obtain $q_1 = q_2 = 0$. The oscillations become

$$\theta_1 = \frac{v}{2}\left[\sin t + \frac{\sin \omega_2 t}{\omega_2}\right] \quad \text{and} \quad \theta_2 = \frac{v}{2}\left[\sin t - \frac{\sin \omega_2 t}{\omega_2}\right].$$

If $k \ll 1$, then $\omega_2 = \sqrt{1+2k} \simeq 1 + k$ and $\dfrac{1}{\omega_2} \simeq 1 - k$. We denote $\varepsilon = \dfrac{\omega_2 - 1}{2} \simeq \dfrac{k}{2}$ and $\Omega = \dfrac{\omega_2 + 1}{2}$; we obtain

$$\theta_1 \simeq v \cos \varepsilon t \sin \Omega t \quad \text{and} \quad \theta_2 \simeq v \sin \varepsilon t \cos \Omega t,$$

where $\Omega \simeq 1$ and the amplitude of $v \cos \varepsilon t$ is very slightly variable. For $\cos \varepsilon t = 0$ corresponding to $t = \dfrac{\pi}{2\varepsilon}$, $\theta_1 = 0$ and θ_2 will be maximum. For $t = \dfrac{\pi}{\varepsilon}$, $\theta_2 = 0$ and θ_1 is maximum. Let us denote $\mathcal{T} = \dfrac{\pi}{2\varepsilon}$, and we obtain the beating phenomenon. In order to observe this phenomenon, we advise the reader to represent the curves of $\theta_1(t)$ and $\theta_2(t)$ on the same graph.

Solution to Exercise 45

1) The kinetic energy is

$$\frac{1}{2} m \left(\dot{x}^2 + \frac{4}{3}\dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}[\dot{x}, \dot{y}, \dot{z}]\, \mathcal{A} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad \text{where} \quad \mathcal{A} = \begin{bmatrix} m & 0 & 0 \\ 0 & \frac{4}{3}m & 0 \\ 0 & 0 & m \end{bmatrix}.$$

2) The potential energy is

$$W = 2\, m\, n^2 \left(x^2 + y^2 + z^2 - x\, y - y\, z\right) = \frac{1}{2}[x, y, z]\, \mathcal{B} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where

$$\mathcal{B} = m\, n^2 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

3) The eigenvalues of $\mathcal{B}$ with respect to $\mathcal{A}$ satisfy

$$m \det \begin{bmatrix} 4n^2 - \lambda & -2n^2 & 0 \\ -2n^2 & 4n^2 - \frac{4}{3}\lambda & -2n^2 \\ 0 & -2n^2 & 4n^2 \end{bmatrix} = \frac{1}{3}\left(4\,n^2\right)\left[4\,\lambda^2 - 28\;n^2 + 24\,n^4\right] = 0.$$

We obtain $\lambda_1 = n^2$, $\lambda_2 = 4\,n^2$ and $\lambda_3 = 6\,n^2$. We can deduce the eigenfrequencies of the system: $\omega_1 = n$, $\omega_2 = 2\,n$ and $\omega_3 = n\sqrt{6}$.

4) The Lagrange equations of motion are

$$\begin{cases} \ddot{x} = -4\,n^2\,x + 2\,n^2\,y, \\ \dfrac{4}{3}\,\ddot{y} = 2\,n^2\,x - 4\,n^2\,y + 2\,n^2\,z, \\ \ddot{z} = 2\,n^2\,y - 4\,n^2\,z. \end{cases}$$

Search the values of ω such that

$$x = A\,\cos\,\omega(t-\tau),\; y = B\,\cos\,\omega(t-\tau),\; z = C\,\cos\,\omega(t-\tau),$$

where A, B, C and τ are constants. The system of Lagrange equations (with the condition that the values of A, B, C are not simultaneously zero) implies that the determinant for $\mathcal{B} - \omega^2\,\mathcal{A}$ is zero. The eigenvectors associated with frequencies are:

$$\begin{cases} \omega^2 = n^2, \quad \text{we obtain the eigenvector} \quad \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \\ \omega^2 = 4\,n^2, \quad \text{we obtain the eigenvector} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \\ \omega^2 = 6\,n^2, \quad \text{we obtain the eigenvector} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \end{cases}$$

We obtain the position of the system for each eigenfrequency:

$$\begin{cases} \omega^2 = n^2, & \text{we obtain the position} \quad {}^{\bullet(M_1)}\ {}^{\bullet(M_2)}\ {}^{\bullet(M_3)}, \\ \omega^2 = 4\,n^2, & \text{we obtain the position} \quad {}^{\bullet(M_1)}\ \bullet(M_2)\ {}_{\bullet(M_3)}, \\ \omega^2 = 6\,n^2, & \text{we obtain the position} \quad {}^{\bullet(M_1)}\ {}_{\bullet(M_2)}\ {}^{\bullet(M_3)}. \end{cases}$$

For a periodic motion, the eigenfrequencies must be commensurable. Since the eigenfrequencies have values $\omega_1 = n,\ \omega_2 = 2\,n$ and $\omega_3 = n\,\sqrt{6}$, the third mode does not come into play.

5) In the basis of eigenvectors, the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ becomes $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, with

$$\begin{cases} x = 2\,\alpha + \beta + \gamma \\ y = 3\,\alpha - \gamma \\ z = 2\,\alpha - \beta + \gamma \end{cases} \iff \begin{cases} \alpha = \dfrac{1}{10}\,(x + 2\,y + z)\,, \\ \beta = \dfrac{1}{2}\,(x - z)\,, \\ \gamma = \dfrac{1}{10}\,(3\,x - 4\,y + 3\,z)\,. \end{cases}$$

Then, the kinetic energy and the potential energy are

$$T = \frac{1}{2}\,m \sum_i \dot{Q}_i^2, \quad W = \frac{1}{2}\,m \sum_i w_i^2\,Q_i^2,$$

with

$$Q_1 = 2\,\sqrt{5}\,\alpha, \quad Q_2 = \sqrt{2}\,\beta, \quad Q_3 = \sqrt{\frac{10}{3}}\,\gamma.$$

6) If the initial conditions are

$$x - a = y - 2a = z - 3a = 0, \quad \dot{x} = u,\ \dot{y} = \dot{z} = 0,$$

then we obtain the solution

$$\begin{cases} x - a = \dfrac{u}{10}\left[\dfrac{2}{\omega_1}\,\sin\,\omega_1\,t + \dfrac{5}{\omega_2}\,\sin\,\omega_2\,t + \dfrac{3}{\omega_3}\,\sin\,\omega_3\,t\right], \\ y - 2a = \dfrac{3\,u}{10}\left[\dfrac{1}{\omega_1}\,\sin\,\omega_1\,t - \dfrac{1}{\omega_3}\,\sin\,\omega_3\,t\right], \\ z - 3a = \dfrac{u}{10}\left[\dfrac{2}{\omega_1}\,\sin\,\omega_1\,t - \dfrac{5}{\omega_2}\,\sin\,\omega_2\,t + \dfrac{3}{\omega_3}\,\sin\,\omega_3\,t\right]. \end{cases}$$

References

Arnold, V.I. (1989). *Mathematical Methods of Classical Mechanics*. Springer, Berlin.

Arnold, V.I. (1992). *Ordinary Differential Equations*. Springer, Berlin.

Béletski, V.V. (1986). *Essais sur le mouvement des corps cosmiques*. Mir, Moscow.

Berest, P. (1997). *Calcul des variations*. Ellipses, Paris.

Bonvalet, M. (1993). *Les principes variationnels*. Masson, Paris.

Brousse, P. (1968). *Mécanique*. Armand Colin, Paris.

Couty, R. and Ezra, J. (1980). *Analyse*. Armand Colin, Paris.

De Broglie, L. (1924). Recherches sur la théorie des quanta. Thesis, Université de Paris.

Germain, P. (1987). *Mécanique*, vols 1 and 2. Ellipses, Paris.

Hand, L.N. and Finch, J.D. (2008). *Analytical Mechanics*. Cambridge University Press, Cambridge.

Kibble, T.W.B. (1973). *Classical Mechanics*. McGraw-Hill, London.

Lagrange, J.-L. (1965). *Mécanique analytique*, vols 1 and 2. Blanchard, Paris.

Lanczos, C. (1970). *The Variational Principles of Mechanics*. Dover Publications Inc., New York.

Luré, L. (1968). *Mécanique analytique*, vols 1 and 2. Masson, Paris.

Martin, P. (1967). *Applications de l'algèbre et de l'analyse à la géométrie*. Armand Colin, Paris.

Pars, L.A. (1979). *A Treatise on Analytical Mechanics*. Ox Bow Press, Woodbridge.

Pérès, J. (1962). *Mécanique générale*. Masson, Paris.

Queysane, M. (1971). *Algèbre*. Armand Colin, Paris.

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